CONSTRUCTION & PHYSICAL APPLICATION OF THE FRACTIONAL CALCULUS ‡

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Introduction. If you knew that

$$
\begin{aligned}
(1+x)^2 &= 1 + 2x + \tfrac{1}{2!}2(2-1)x^2\\
(1+x)^3 &= 1 + 3x + \tfrac{1}{2!}3(3-1)x^2 + \tfrac{1}{3!}3(3-1)(3-2)x^3\\
&\;\;\vdots\\
(1+x)^p &= 1 + \sum_{k=1} \tfrac{1}{k!}p(p-1)(p-2)\cdots(p-[k-1])x^k \qquad (1)
\end{aligned}
$$

but were unaware of Newton's discovery that—subject to restrictions on the value of x, and even though the series may fail to terminate—(1) works for *all* real values of p (not necessarily an integer), you would labor under a severe handicap, and would have to exercise some cleverness to establish even so simple a result as

$$\frac{d}{dx}x^{\frac{1}{2}} = \tfrac{1}{2}x^{-\frac{1}{2}}$$

Similarly and relatedly, if you possessed detailed knowledge of the properties of

$$n! \equiv n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1$$

and its cognates (such, most notably, as $\binom{n}{m} \equiv n!/m!(n-m)!$) but remained ignorant of Euler's wonderful invention

$$
\begin{aligned}
\Gamma(z) &\equiv \int_0^\infty t^{z-1}e^{-t}\,dt\\
&= s^z\cdot\underbrace{\int_0^\infty e^{-st}t^{z-1}\,dt}_{} \qquad : \quad \Re[z]>0,\ \Re[s]>0\\
&\qquad\qquad = \text{Laplace transform of the } z\text{-parameterized function } t^{z-1}\\
&= \lim_{n\uparrow\infty}\frac{n!n^z}{z(z+1)\cdots(z+n)} \qquad : \quad z \neq 0,-1,-2,\ldots
\end{aligned}
$$

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and of its properties—most notably

$$\Gamma(z+1) = z\Gamma(z) = z! \quad : \quad z = 0, 1, 2, \ldots$$

—you would labor at a distinct disadvantage, cut of from many/most of the inexhaustible resources of higher analysis.

My thesis today is that, however secure your command of the differential and integral calculus of the textbooks, if your are unable to assign useful meaning to (for example) the expression

$$\frac{d^{\frac{1}{2}}f(x)}{dx^{\frac{1}{2}}}$$

or, more generally, to $d^p f(x)/dx^p$ where p is "any" number (positive or negative, real or complex), then you labor deprived of a powerful resource.

Early in the present century physicists learned, under the leadership of people like P. W. Debye and A. Sommerfeld, to escape the "tyranny of the real line," to do their physics on the complex plane, and only at the end of their calculations to let z become real. More recently they have learned to escape the "tyranny of dimensional integrality;" though fractal dimension comes instantly to mind, I am thinking here more particularly of the many papers (especially papers treating statistical mechanical and field-theoretic topics) that adhere to the pattern

$$\begin{aligned}&\big\{\text{difficult physics in 3 dimensions}\big\}\\ &\qquad = \lim_{\epsilon\to 0}\big\{\text{relatively easy physics in } 3-\epsilon \text{ dimensions}\big\}\end{aligned}$$

The developments I shall be describing are similar in spirit and motivating intent. They permit one to perform with ease computations which would otherwise prove difficult, to formulate concepts and distinctions which would otherwise remain elusive. They can be symbolized

$$\Big(\frac{d}{dx}\Big)^{\pm\,\text{integer}} \longrightarrow \Big(\frac{d}{dx}\Big)^{\text{real or complex}}$$

We recall in this connection that

$$\Big(\frac{d}{dx}\Big)^{\text{integer}} \longrightarrow \Big(\frac{d}{dx}\Big)^{\pm\,\text{integer}}$$

was achieved already by the Fundamental Theorem of the Calculus, and that

$$(1+x)^{\text{integer}} \longrightarrow (1+x)^{\text{real or complex}}$$

entailed invention of the concept of infinite series; in the latter context as in the context that will concern us, a kind of "interpolation" is going on, but it is interpolation *in the exponent*. In both contexts, infinity intrudes. Given a "seed" $f(x)$ subject to appropriate restrictions, we might employ operations of the ordinary calculus to construct

$$\cdots \leftarrow \iiint f \leftarrow \iint f \leftarrow \int f \leftarrow f \rightarrow f' \rightarrow f'' \rightarrow f''' \rightarrow \cdots$$

and seek to "interpolate" between the points thus marked out in function space. An element of ambiguity attaches, of course, to all interpolation/extrapolation schemes; one looks for the scheme that is most "natural" in the sense "most empowering." It is in that always-somewhat-vague sense that (for example) one defends the claim that $\Gamma(n+1)$ provides the "most natural" interpolation amongst the discrete numbers $n!$.[1] In just that same sense, one develops the strong conviction—but cannot explicitly prove—that the fractional calculus does in fact proceed optimally to its interpolative goal.

The vision of a "fractional calculus" was evident already to the founding fathers of the ordinary calculus (which—but for the double meaning of a word which becomes intolerable in this context—we are tempted to call the "integral calculus"). Leibniz—who was, after all, the inventor of both the $d^n f/dx^n$ notation and of $\int f(x)\,dx$ —wrote in September of 1695 to his friend l'Hospital as follows:[2]

> *"Jean Bernoulli seems to have told you of my having mentioned to him a marvelous analogy which makes it possible to say in a way that successive differentials are in geometric progression. One can ask what would be a differential having as its exponent a fraction. You see that the result can be expressed by an infinite series, although this seems removed from Geometry, which does not yet know of such fractional exponents. It appears that one day these paradoxes will yield useful consequences, since there is hardly a paradox without utility. Thoughts that mattered little in themselves may give occasion to more beautiful ones."*

Thirty-five years later, Euler expressed a similar thought, and took explicit note of the fact that a kind of interpolation theory comes necessarily into play:

> *"Concerning transcendental progressions whose terms cannot be given algebraically: when n is a positive integer, the ratio $d^n f/dx^n$ can always be expressed algebraically. Now it is asked: what kind of*

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[1] See, in this connection, R. Remmert, "Wieland's theorem about the Γ function," Amer. Math. Monthly **103**, 214 (1996). It is, according to the author, "well known" that $\Gamma(1) = 1$ and $\Gamma(z+1) = z\Gamma(z)$ do not in themselves serve to characterize $\Gamma(z)$. A elegant side condition sufficient to the purpose was described by H. Bohr & J. Mollerup in 1922. He draws attention to an alternative—and equally beautiful—side condition reported by H. Wielandt in 1939.

[2] My source here is B. Mandelbrot (*Fractals: Form, Chance, and Dimension* (1977), p. 299), who claims responsibility for the translation.

> *ratio can be made if n be a fraction? . . . the matter may be expedited with the help of the interpolation of series, as explained earlier in this dissertation."*

. . . but I do not know the identity of the "dissertation" to which he refers; the notion of a "fractional calculus" is, so far as I am aware, not mentioned in his monumental *Institutiones calculi differentialis* of 1755, and first public mention of the so-called "Euler integrals"

$$
\begin{aligned}
\Gamma(p) &= \int_0^\infty x^{p-1}e^{-x}\,dx \\
B(m,n) &= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}
\end{aligned}
$$

—which play such a central role in this story—did not appear until publication of *Institutiones calculi integralis* (3 volumes, 1768–1770). Notable contributions to the field were made successively by Laplace, Fourier, Abel, Liouville, Riemann, Heaviside and—in the present century—by Bateman, Hardy, Weyl, Riesz and Courant, as well as by many pure and applied mathematicians of lesser reknown. Our subject can claim, therefore, to be well upwards of 200 years old, and its foundations have been securely in place for more than a century. Yet the first book-length account of the field did not appear until 1974, when Keith B. Oldham & Jerome Spanier published *The Fractional Calculus: Theory & Applications of Differentiation & Integration to Arbitrary Order.*[3] Though it retains its place as the primary reference in the field, this monograph was joined recently by K. S. Miller & B. Ross' *An Introduction to the Fractional Calculus and Fractional Differential Equations* (1993). Both books are very accessibly written, and so far as concerns matters of design and emphasis they are neatly complementary; when you undertake study of this field you will want to have both within easy reach. Both books provide brief but usefully detailed accounts of the history of the fractional calculus (as well as valuable bibliographic data), and it is from them that, in the absence of explicit indication to the contrary, my own historical remarks have been taken.

1. Elementary preliminaries. Let us agree (pending future refinements) to write

$$D \equiv \frac{d}{dx}$$

By the Fundamental Theorem of the Calculus

$$D\int_a^x f(\xi)\,d\xi = f(x) \tag{1}$$

[3] Oldham is a professor of chemistry at Trent University in Peterborough, Ontario; Spanier is—less surprisingly—a professor of mathematics at Claremont Graduate School. They are joint authors also of the valuable handbook *An Atlas of Functions* (1987).

irrespective of the constant value assigned to a. Moreover

$$\int_a^x Df(\xi)\,d\xi = f(x) - f(a) \tag{2}$$

These familiar statements suggest a range of meanings that might with equal plausibility be assigned to the operator D^{-1}. I will call a the "fiducial point,"[4] and write

$${}_aD_x^{-1}f(x) \equiv \int_a^x f(\xi)\,d\xi \tag{3}$$

We then have

$$D \cdot {}_aD_x^{-1}f(x) = f(x) \tag{4.1}$$

$${}_aD_x^{-1} \cdot D\,f(x) = f(x) - f(a) \tag{4.2}$$

Evidently ${}_aD_x^{-1}$ is unrestrictedly a right inverse of D, but is a left inverse only with respect to functions that vanish at the fiducial point: $f(a) = a$. Only with respect to such specialized functions do D and ${}_aD_x^{-1}$ commute. Note that the operator ${}_aD_x^{-1}$ is *non-local but linear*; infinitely much $f(x)$-data is absorbed into its defining action. Note also that $F(x) \equiv {}_aD_x^{-1}f(x)$ has the property that it automatically vanishes at the fiducial point: $F(a) = 0$. By extension of (2) we have

$$\left.\begin{aligned} {}_aD_x^{-2}f(x) &\equiv \int_a^x\!\!\int_a^{\xi'} f(\xi)\,d\xi d\xi' \\ {}_aD_x^{-3}f(x) &\equiv \int_a^x\!\!\int_a^{\xi''}\!\!\int_a^{\xi'} f(\xi)\,d\xi d\xi' d\xi'' \\ &\vdots \end{aligned}\right\} \tag{5}$$

These operators all feed—each in its own characteristic way—on the same infinite set of $f(x)$-data.

To discuss the higher order implications of (4) I adopt (as frequently I shall in what follows) an abbreviated notation

$${}_aD_x^{-m} \quad \rightarrowtail \quad D^{-m} \quad \text{when no confusion can result}$$

Taking D^p as our "seed" and proceeding recursively with the aid of (4.1) we obtain

$$\begin{aligned} D^p f = D^p(D \cdot D^{-1})f &= D^{p+1} \cdot D^{-1}f = D^{p+1}(D \cdot D^{-1})D^{-1}f \\ &= D^{p+2} \cdot D^{-2}f \\ &\;\;\vdots \\ &= D^{p+n} \cdot D^{-n}f \end{aligned}$$

[4] Later we will have reason to set $a = 0$ (Riemann–Liouville), else $-\infty$ (Liouville), else $+\infty$ (Weyl), but it will in general be our practice to leave the fiducial point unspecified until we encounter some specific reason to do otherwise; ultimately we will, in one specialized connection, allow a (in the sense $\lim_{a\uparrow x}$) to join the variables of the theory.

for all (sufficiently differentiable) functions $f(x)$. By an identical argument

$$D^{-p}f = D^m \cdot D^{-p-m}f$$

Evidently composit operators of the types

$$\left.\begin{array}{c}\text{differentiation}\cdot\text{differentiation}\\ \text{differentiation}\cdot\text{integration}\\ \text{integration}\cdot\text{integration}\end{array}\right\}\quad\text{obey the LAW OF EXPONENTS}$$

in this sense:

$$\left.\begin{array}{r}(\text{differentiation})^m\cdot(\text{differentiation})^n = (\text{differentiation})^{m+n}\\ (\text{differentiation})^m\cdot(\text{integration})^n = \begin{cases}(\text{differentiation})^{m-n} & \text{if } m\geq n\\ (\text{integration})^{n-m} & \text{if } m\leq n\end{cases}\\ (\text{integration})^m\cdot(\text{integration})^n = (\text{integration})^{m+n}\end{array}\right\}\tag{6.1}$$

Operators of the type $(\text{integration})^m\cdot(\text{differentiation})^n$ are, however, more complicated, as was evident already at (4.2) and as the following examples serve to illustrate:

$$\begin{aligned}
D^{-1}D^3 f &= D^{+2}f - f''(a)\\
D^{-1}D^2 f &= D^{+1}f - f'\,(a)\\
D^{-1}D^1 f &= D^{+0}f - f\;\,(a)\\[1em]
D^{-2}D^3 f &= D^{+1}f - f''(a)(x-a) - f'(a)\\
D^{-2}D^2 f &= D^{+0}f - f'\,(a)(x-a) - f(a)\\
D^{-2}D^1 f &= D^{-1}f - f\;\,(a)(x-a)\\[1em]
D^{-3}D^3 f &= D^{+0}f - \tfrac{1}{2}f''(a)(x-a)^2 - f'(a)(x-a) - f(a)\\
D^{-3}D^2 f &= D^{-1}f - \tfrac{1}{2}f'\,(a)(x-a)^2 - f\,(a)(x-a)\\
D^{-3}D^1 f &= D^{-2}f - \tfrac{1}{2}f\;\,(a)(x-a)^2
\end{aligned}\tag{7}$$

In general, one has

$$\begin{aligned}&(\text{integration})^n\cdot(\text{differentiation})^m\\ &\qquad= \begin{cases}(\text{differentiation})^{m-n} & +\text{ extra terms if } m\geq n\\ (\text{integration})^{n-m} & +\text{ extra terms if } m\leq n\end{cases}\end{aligned}\tag{6.2}$$

and to kill the extra terms must constrain $f(x)$ to satisfy conditions of the form $f(a) = f'(a) = f''(a) = \cdots = 0$. Similar complications arise typically when one looks to operators of mixed type $\cdots D^m D^{-n} D^p D^{-q}\cdots$ but so, on occasion, do some surprising simplifications; look, for example, to the case

$$D^5 D^{-2}\underline{D^3 D^{-2}}\,f = \underline{D^5 D^{-2}}D^1 f = D^3 D^1 f = D^4 f$$

Evidently $D^5D^{-2}D^3D^{-2} = D^{5-2+3-2}$. On the other hand,

$$\underline{D^1D^{-2}}\ \underline{D^5D^{-2}}\, f = D^{-1}D^3 f = D^2 f + \text{extra terms}$$

so $D^1D^{-2}D^3D^{-2} \neq D^{1-2+3-2}$. The simple facts reported above—which will acquire deeper significance as we move farther into our subject—reflect the deep asymmetry which is present already in (4), and is reflected also in the statements

$$\begin{array}{ll}
\vdots & \\
D^{+3}f(x) = f'''(a) & \text{at } x = a \\
D^{+2}f(x) = f''\ (a) & \text{at } x = a \\
D^{+1}f(x) = f'\ \ (a) & \text{at } x = a \\
D^{\pm 0}f(x) = f\ \ \ (a) & \text{at } x = a \\
D^{-1}f(x) = 0 & \text{at } x = a \\
D^{-2}f(x) = 0 & \text{at } x = a \\
D^{-3}f(x) = 0 & \text{at } x = a \\
\vdots &
\end{array}$$

The asymmetry traces ultimately to the circumstance that

$$\begin{array}{l}
(\text{differentiation})^{\text{integer}} \ \ \text{is a } \textsc{local} \text{ operator} \\
(\text{integration})^{\text{integer}} \ \text{is a } \textsc{nonlocal} \text{ operator}
\end{array}$$

It will emerge that, within the fractional calculus, (differentiation)^p is more "integration-like" than" differentiation-like," in this important sense:

$$(\text{differentiation})^p \text{ is } \textsc{local only exceptionally}, \text{ namely at } p = 0, 1, 2, \ldots$$

just as (and for essentially the same reason that) the expansion of $(1+x)^p$ terminates if and only if $p = 0, 1, 2, \ldots$

I digress to remark that what I have called "extra terms" are familiar to physicists as "transient terms"—terms which enter additively into the solutions of linear differential equations, where they serve to accommodate initial data but (in typical applications) die exponentially. Look, for example, to the problem

$$\ddot{f}(t) = g(t) \quad : \quad g(t) \text{ given; } f(0) \text{ and } \dot{f}(0) \text{ stipulated}$$

Multiplication by $D^{-2} \equiv {}_0D_t^{-2}$ gives (I make use here of (7))

$$D^{-2}D^2 f = D^0 f - f(0) - \dot{f}(0)t = D^{-2}g$$

In other words

$$f(t) = \underbrace{f(0) + \dot{f}(0)t}_{=\text{ "transient terms," which on this occasion don't actually die}} + \int_0^t\!\!\int_0^\tau g(\tau')\,d\tau' d\tau$$

The first substantive step toward the creation of a fractional calculus was taken in 1819 when S. F. Lacroix—quite casually, and with no evident practical intent—remarked that the familiar formula

$$D^m x^p = p(p-1)(p-2)\cdots(p-m+1)x^{p-m}$$

—which we notate $D^m x^p = \frac{p!}{(p-m)!}\,x^{p-m}$ when p is an integer—can in every case be notated

$$D^m x^p = \frac{\Gamma(p+1)}{\Gamma(p-m+1)}\,x^{p-m} \tag{8.1}$$

and that (8.1) *makes formal sense even when m is not an integer.* The fact that

$$D^m x^p = 0 \quad \text{when } m \text{ and } p \text{ are integers with } m > p$$

can then be attributed to the circumstance that $\Gamma(0), \Gamma(-1), \Gamma(-2), \ldots$ are singular. Proceeding in the other direction, one has

$$\begin{aligned}
D^{-1}x^p &\equiv {}_0D_x^{-1}x^p \equiv \int_0^x \xi^p\,d\xi = \tfrac{1}{(p+1)}x^{p+1}\\
D^{-2}x^p &= \tfrac{1}{(p+2)(p+1)}x^{p+2}\\
&\ \vdots\\
D^{-n}x^p &= \tfrac{1}{(p+n)\cdots(p+2)(p+1)}x^{p+n} = \tfrac{p!}{(p+n)!}x^{p+n}
\end{aligned}$$

which can in the same spirit be written

$$D^{-n}x^p = \frac{\Gamma(p+1)}{\Gamma(p+n+1)}\,x^{p+n} \tag{8.2}$$

We note that $m \rightleftharpoons -n$ sends $(8.1) \rightleftharpoons (8.2)$, and from

$$\begin{aligned}
D^m D^{-n}x^p &= \frac{\Gamma(p+n+1)}{\Gamma(p+n-m+1)}\frac{\Gamma(p+1)}{\Gamma(p+n+1)}x^{p+n-m}\\
&= \frac{\Gamma(p+1)}{\Gamma(p-(m-n)+1)}x^{p-(m-n)}\\
&= D^{m-n}x^p
\end{aligned} \tag{9}$$

conclude that Lacroix' construction supports an unrestricted law of exponents. Lacroix found himself in position, therefore, to assign a formally very simple (if computationally intricate) meaning to expressions of the type

$$D^\mu\Big\{\sum_p f_p x^p\Big\} \quad : \quad \mu \text{ any number, real or complex}$$

and did not fail to note that such a calculus would give surprising results even in the simplest cases; one has, for example, the "semiderivatives"

$$D^{\frac{1}{2}}x = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}x^{\frac{1}{2}} = 2\sqrt{\frac{x}{\pi}}$$

and, perhaps more remarkably,

$$D^{\frac{1}{2}}x^0 = \frac{\Gamma(1)}{\Gamma(\frac{1}{2})}x^{-\frac{1}{2}} = \frac{1}{\sqrt{\pi}}x^{-\frac{1}{2}} = \sqrt{\frac{1}{\pi x}} \tag{10}$$

Lacroix' construction (which subsumes all of ordinary calculus) survives as a sub-calculus within the full-blown fractional calculus. We note with interest that the interpolative burden of the construction is borne by Euler's Γ function. We will, when we turn to applications, have particular and repeated need of (10), which is so typical of the field that it deserves to be embroidered onto the banner carried by fractional revolutionaries.

I bring these introductory remarks to a close with mention of the fact that Fourier, in 1822, had occasion to introduce

$$\frac{d^m}{dx^m}\cos p(x-a) = p^m \cos\big[p(x-a) + \tfrac{1}{2}m\pi\big]$$

into

$$f(x) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(\alpha)\,d\alpha \int_{-\infty}^{+\infty} \cos p(x-a)\,dp$$

to obtain (after the notational adjustment $m \longrightarrow \mu$)

$$\frac{d^\mu}{dx^\mu}f(x) = \tfrac{1}{2\pi}\int_{-\infty}^{+\infty} f(\alpha)\,d\alpha \int_{-\infty}^{+\infty} p^\mu \cos\big[p(x-a) + \tfrac{1}{2}\mu\pi\big]\,dp$$

and in that connection to observe that "the number μ... will be regarded as any quantity whatsoever, positive or negative." But Fourier seems not to have pursued the implications of his throw-away remark, which remain, so far as I am aware, largely unexplored; in the modern literature one encounters the Laplace transform often, but the Fourier transform only seldom.

2. Grünwald's construction. The operators ${}_aD_x^{-1}$ are "lefthanded;" we adopt, therefore, a lefthanded definition of the ordinary derivative

$$Df(x) \equiv \lim_{h\downarrow 0}\frac{f(x) - f(x-h)}{h}$$

which can, by Taylor's theorem $e^{\alpha D}f(x) = f(x+\alpha)$, be notated

$$Df(x) = \lim_{h\downarrow 0}\frac{1-e^{-hD}}{h}f(x) \quad : \quad \text{all nice functions } f(x) \tag{11}$$

It becomes in this light natural to write

$$\begin{aligned} D^m &\equiv \lim_{h\downarrow 0}\left(\frac{1-e^{-hD}}{h}\right)^m \quad : \quad m = 0,1,2,\ldots \\ &= \lim_{h\downarrow 0}\frac{1}{h^m}\Big\{1 - me^{-hD} + \tfrac{1}{2!}m(m-1)e^{-2hD} - \cdots\Big\} \end{aligned} \tag{12}$$

from which we recover in a unified way the familiar results

$$\left.\begin{aligned} D^2 f(x) &= \lim_{h\downarrow 0} \tfrac{f(x)-2f(x-h)+f(x-2h)}{h^2}\\ &= \lim_{h\downarrow 0} \tfrac{[f(x)-f(x-h)]-[f(x-h)-f(x-2h)]}{h^2}\\ D^3 f(x) &= \lim_{h\downarrow 0} \tfrac{f(x)-3f(x-h)+3f(x-2h)-f(x-4h)}{h^3}\\ D^4 f(x) &= \lim_{h\downarrow 0} \tfrac{f(x)-4f(x-h)+6f(x-2h)-4(x-3h)+f(x-4h)}{h^4}\\ &\vdots \end{aligned}\right\} \tag{13}$$

The operators D^m, as (13) makes explicitly clear, feed on finite $f(x)$-data sets of ascending size, and are in this sense "local" operators.

In view of the structure of (12) it becomes entirely natural to relax the requirement that m be an integer, writing (for example)

$$D^{\frac{1}{2}} = \lim_{h\downarrow 0} \frac{1}{\sqrt{h}}\Big\{1 - \tfrac{1}{2}e^{-hD} + \tfrac{1}{8}e^{-2hD} + \cdots\Big\} \tag{14}$$

and

$$D^{-1} = \lim_{h\downarrow 0} \Big\{1 + e^{-hD} + e^{-2hD} + e^{-3hD} \cdots\Big\}h \tag{15}$$

Elaborating on (15), we have

$$\begin{aligned} D^{-1} f(x) &= \lim_{h\downarrow 0} \sum_{k=0}^{\infty} e^{-khD} f(x)\cdot h \qquad\qquad (16)\\ &= \lim_{h\downarrow 0} \Big\{\sum_{k=0}^{\infty} f(x-kh)\cdot h\Big\}\\ &= \int_{-\infty}^{x} f(\xi)\,d\xi\\ &= {}_{-\infty}D_x^{-1} f(x) \quad \text{in the notation of (3)} \end{aligned}$$

A Riemann integral has been spontaneously associated with the meaning of D^{-1}, but the fiducial point has with equal spontaneity been placed at $a = -\infty$. The adjustments which would serve to place the fiducial point at an arbitrary point a are, however, pretty evident; backing up to (16), we set $h = (x-a)/N$ and in place of taking $h \downarrow 0$ take $N \uparrow \infty$, writing

$$\begin{aligned} {}_aD_x^{-1} f(x) &= \int_a^x f(\xi)\,d\xi\\ &= \lim_{N\uparrow\infty} \Big\{\sum_{k=0}^{N-1} f\Big(x - k\frac{x-a}{N}\Big)\cdot\frac{x-a}{N}\Big\} \end{aligned} \tag{17}$$

It seems natural to require that the adjustment (16)→(17) pertain, if to D^{-1}, then to *all* the D^m operators. To that end, we return to (36) but keep only the first N terms, writing

$$ {}_aD_x^m \sim \frac{1}{h^m}\sum_{k=0}^{N-1}(-)^k\binom{m}{k}e^{-khD}\bigg|_{h=(x-a)/N} $$

where the $h \to (x-a)/N$ is delayed because we don't want the D-operators to sense the x-dependence of h. Thus do we obtain

$$ {}_aD_x^m f(x) = \lim_{N\uparrow\infty}\Big[\frac{N}{x-a}\Big]^m\sum_{k=0}^{N-1}(-)^k\binom{m}{k}f\Big(x-k\frac{x-a}{N}\Big) \tag{18} $$

At $m=1$ the preceding formula, for all of its seeming complexity, yields the simple result

$$ {}_aD_x^1 f(x) = \lim_{N\uparrow\infty}\frac{f(x)-f(x-\frac{x-a}{N})}{\frac{x-a}{N}} $$

which clearly reproduces the standard definition of the first derivative. At $m=-1$ we have $(-)^k\binom{-1}{k}=+1$ (all k) and (by explicit design) recover

$$ {}_aD_x^{-1} f(x) = \int_a^x f(\xi)\,d\xi $$

When we insert

$$ \binom{m}{k} = \frac{\Gamma(m+1)}{\Gamma(k+1)\Gamma(m+1-k)} $$

into (18) we obtain an equation

$$ {}_aD_x^m f(x) \equiv \lim_{N\uparrow\infty}\Big[\frac{N}{x-a}\Big]^m\sum_{k=0}^{N-1}(-)^k\frac{\Gamma(m+1)}{\Gamma(k+1)\Gamma(m+1-k)} \cdot f\Big(x-k\frac{x-a}{N}\Big) \tag{19.1} $$

which assigns natural meaning to the *derivative operators* ${}_aD_x^\mu$ *of all (integral/non-integral non-negative) orders* $\mu \geq 0$.[5] If m is a negative integer, then $n \equiv -m$ is a positive integer, and we have

$$ \begin{aligned} (-)^k\binom{m}{k} &= \frac{n(n+1)(n+2)\cdots(n+k-1)}{k!} \\ &= \frac{(n+k-1)!}{k!(n-1)!} = \binom{n+k-1}{k} \\ &= \frac{\Gamma(k+n)}{\Gamma(k+1)\Gamma(n)} \end{aligned} $$

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[5] It will be my practice to make replacements of the form $m \rightarrowtail \mu, n \rightarrowtail \nu$ when I want to emphasize that an integrality assumption has been abandoned.

Returning with this result to (18) we obtain

$$ {}_aD_x^{-n}f(x) \equiv \lim_{N\uparrow\infty}\Big[\frac{x-a}{N}\Big]^n \frac{1}{\Gamma(n)}\sum_{k=0}^{N-1}\frac{\Gamma(k+n)}{\Gamma(k+1)}f\Big(x-k\frac{x-a}{N}\Big) \tag{19.2} $$

which serves by $n \rightarrowtail \nu$ to assign natural meaning to the concept of a *"fractional integration operator"* ${}_aD_x^{-\nu}$ *of arbitrary (positive) order* $\nu > 0$.

The definitions (19) are precisely the definitions put forward in §2.2 and §3.2 of Oldham & Spanier, where they are attributed to A. K. Grünwald (1867)[6] and E. L. Post,[7] and held to be "fundamental in that they involve the fewest restrictions on the functions to which they apply; [moreover, they]... avoid explicit use of the ordinary derivative and integral." Grünwald's construction is presented in Chapter II, §7 of Miller & Ross where it is accompanied by no such claim, though those authors do remark that "Grünwald's definition is very appealing in that it makes no assumptions other than that $f(x)$ be defined. On the negative side, it is very difficult to calculate the limit in concrete cases. *En revanche* it has the virtue... that it may be used to calculate approximately the fractional derivative." Neither pair of authors attempts to reconstruct Grünwald's motivating argument, which I gather from a remark of Post's must have differed only cosmetically from my own.

It is a notable implication of (19) that the action of what Oldham & Spanier (rather unfelicitously, in my view) call the "differintegration" operators

$$ {}_aD_x^{\lambda} \text{ are non-local except only in the cases } \lambda = 0, 1, 2, 3, \ldots $$

Those operators feed in every case on the same $f(x)$-data, which however they weight in distinct ways—very *simple* distinct ways, as will soon emerge.

3. The Riemann-Liouville construction. In the case $n = 2$ the definition (19.2) gives

$$ {}_aD_x^{-2}f(x) \equiv \lim_{N\uparrow\infty}\Big[\frac{x-a}{N}\Big]^2 \sum_{k=0}^{N-1}(k+1)f\Big(x-k\frac{x-a}{N}\Big) \tag{20} $$

where we have used $\Gamma(2) = 1$ and $\Gamma(k+2)/\Gamma(k+1) = k+1$. The question arises: How does (20) relate to the

$$ {}_aD_x^{-2}f(x) = \int_0^x\!\int_0^{\xi'} f(\xi)\,d\xi d\xi' \tag{21} $$

[6] "Uber 'begrenzte' Derivationen und deren Anwendung," Zeitschrift für Mathematik und Physik **12**, 441.

[7] "Generalized differentiation," Trans. Amer. Math. Soc. **32**, 723–781 (1930). From Post I gather that Grünwald's original argument was rather clumsy, but so also (in my view) is Post's, at least in many of its notational respects; Post's paper does, however, contain much good material, particularly as relates to the contour integal and Laplace transform aspects of the fractional calculus. He is at pains also to establish contact with the operator methods of Heaviside, Bromwich and Carson.

which was advocated at (5)? By allowing myself—at risk of confusion—to place identical marks on the left sides of the equalities in (20) and (21) I have acquired an obligation to establish that their right sides are equivalent. As, indeed, they are; the argument will, however, produce yet a third way of describing ${}_aD_x^{-2}f(x)$—and, more generally, of describing fractional integrals ${}_aD_x^{-\nu}f(x)$—which is computationally much more advantageous than Grünwald's (19.2), and which provides in fact the practical foundation of the fractional calculus. We will be led thus to the view that

- fractional integrals are *integral transforms* of a specialized type, and that
- fractional derivatives are *ordinary derivatives of fractional integrals.*

Pretty evidently, the right side of (20) is speaking to us about a single integral. We stand in need, therefore, of a mechanism for expressing (certain) iterated integrals as simple integrals. We proceed[8] from the elementary observation that

$$\frac{d}{dx}\int_a^x w(x,y)f(y)\,dy = w(x,x)f(x) + \int_a^x \frac{\partial w(x,y)}{\partial x}f(y)\,dy$$

If we require

$$w(x,x) = 0 \quad \text{and} \quad \frac{\partial w(x,y)}{\partial x} = 1$$

—which is to say: if we set

$$w(x,y) = x - y$$

—then we obtain

$$\frac{d}{dx}\int_a^x (x-y)f(y)\,dy = \int_a^x f(y)\,dy$$

Enlarging upon this pretty result, we have

$$\begin{aligned}
\frac{d^2}{dx^2}\int_a^x (x-y)^2 f(y)\,dy &= \frac{d}{dx}\int_a^x 2(x-y)f(y)\,dy \\
&= 2\int_a^x f(y)\,dy \\
&\ \ \vdots \\
\frac{d^n}{dx^n}\int_a^x (x-y)^n f(y)\,dy &= n!\int_a^x f(y)\,dy \quad : \quad n = 0,1,2,\ldots
\end{aligned} \tag{22.1}$$

[8] My sources are Oldham & Spanier, §2.7 and R. Courant, *Differential & Integral Calculus* (1936), Volume II, p. 221. Courant's, by the way, is the only text known to me that even mentions the existence of a fractional calculus; see his Chapter IV, §7. The topic is not mentioned in Volume I of R. Courant & D. Hilbert, *Methods of Mathematical Physics* (1962), but is mentioned twice in their Volume II; specific citations will be given later.

from which it follows easily that[9]

$$
\begin{aligned}
\int_a^x\!\!\int_a^{y'} f(y)\,dydy' &= \int_a^x (x-y)\;\, f(y)\,dy\\
\int_a^x\!\!\int_a^{y''}\!\!\int_a^{y'} f(y)\,dydy'dy'' &= \tfrac{1}{2!}\int_a^x (x-y)^2 f(y)\,dy\\
&\;\;\vdots\\
\underbrace{\int_a^x\!\!\int_a^{y_n}\!\!\int_a^{y_{n-1}}\!\!\cdots\int_a^{y_2}}_{n\text{-fold iterated}} f(y_1)\,dy_1dy_2\cdots dy_n &= \tfrac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)\,dy
\end{aligned}
\tag{22.2}
$$

which Oldham & Spanier attribute to Cauchy. Bringing (22.2) to (5), we have

$$
{}_aD_x^{-n}f(x) = \tfrac{1}{(n-1)!}\int_0^x (x-y)^{n-1} f(y)\,dy \quad : \quad n = 1,2,3,\ldots \tag{23}
$$

Upon (23) hangs the computational essence of the fractional calculus—as will emerge. But the question immediately before us—assured, as we now are, that (5) $\Longleftrightarrow$ (23)—is this: How does (20) relate to (23)? The answer is that they are *identical*, as I now show. Write

$$
y_k \equiv x-(k+1)\frac{x-a}{N} = \begin{cases} (1-\frac{1}{N})x+\frac{a}{N}\sim x & \text{at } k=0\\ a & \text{at } k=N-1\end{cases}
$$

Then $x-y_k=(k+1)(x-a)/N$, $\Delta y\equiv y_{k+1}-y_k = -(x-a)/N$ and (20) can be notated

$$
\begin{aligned}
{}_aD_x^{-2}f(x) &= \lim_{N\uparrow\infty}\sum_{k=0}^{N-1}(x-y_k)f\Big(y_k-\frac{x}{N}\Big)(-\Delta y)\\
&= +\int_0^x (x-y)f(y)\,dy
\end{aligned}
$$

which is the result claimed. Generalization to the cases $m = 2,3,\ldots$ poses no real difficulty. We are in position therefore to make these simultaneous assertions:

$$
\begin{aligned}
{}_aD_x^{-n}f(x) &= \int_a^x\!\!\int_a^{y_n}\!\!\int_a^{y_{n-1}}\!\!\cdots\int_a^{y_2} f(y_1)\,dy_1dy_2\cdots dy_n \quad : \quad n=1,2,3,\ldots\\
&= \lim_{N\uparrow\infty}\Big[\frac{x-a}{N}\Big]^n\frac{1}{\Gamma(n)}\sum_{k=0}^{N-1}\frac{\Gamma(k+n)}{\Gamma(k+1)}f\Big(x-k\frac{x-a}{N}\Big)\\
&= \frac{1}{\Gamma(n)}\int_a^x (x-y)^{n-1}f(y)\,dy
\end{aligned}
$$

[9] Compare Gradshteyn & Ryzhik **4.631**, p. 620.

The first of those equations is meaningless except when n is an integer. The remaining two equations share, however, the property that they retain formal meaning when the integrality assumption is relaxed: $n \rightarrowtail \nu$. So we allow ourselves tentatively to write

$$
\begin{aligned}
{}_aD_x^{-\nu} f(x) &= \lim_{N\uparrow\infty} \Big[\frac{x-a}{N}\Big]^\nu \frac{1}{\Gamma(\nu)} \sum_{k=0}^{N-1} \frac{\Gamma(k+\nu)}{\Gamma(k+1)} f\Big(x - k\frac{x-a}{N}\Big) \qquad (24.1)\\
&= \frac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} f(y)\, dy \qquad (24.2)
\end{aligned}
$$

Concerning that use of the word "tentatively:" From the statement

$$F(z) = G(z) : z = 1, 2, 3, \ldots \quad \text{and both generalize "naturally"}$$

it does *not* follow that $F(z) = G(z)$: all z; this is the "interpolative ambiguity" problem mentioned previously. In a more careful account of the fractional calculus one would have to describe the conditions—conditions on the structure of $f(x)$—under which the right sides of (24) are in fact equal. This is work which I am happy to leave to the mathematicians;[10] as a physicist, I know myself to be protected from major *faux pas* by the well-constructedness of Nature; it is my habit to look closely to my informal tools only when they have led me to an implausible result. In practice, (24.1) and (24.2) seldom lead to outright contradiction for the simple reason that (24.1) is, except in trivial cases, computationally unworkable. In practice, one usually treats (24.2) as a stand-alone *definition*:

$$ {}_aD_x^{-\nu} f(x) \equiv \frac{1}{\Gamma(\nu)} \int_a^x (x-y)^{\nu-1} f(y)\, dy \quad : \quad \nu > 0 \qquad (25)$$

The Riemann-Liouville construction (25) supplies the foundation of a theory of fractional integration, but yields nonsense at $\nu = 0, -1, -2, \ldots$ These, curiously, are precisely the points at which the expression on the left speaks of the most unexceptionably commonplace objects in the calculus: the derivatives of integral order.[11]

[10] See, for example, Oldham & Spanier, §3.3.

[11] To write

$$D^m f(x) = \frac{1}{\Gamma(-m)} \int_0^x \frac{1}{(x-y)^{m+1}} f(y)\, dy$$

is, however, to be reminded of

$$\frac{d^m f(z)}{dz^m} = \frac{m!}{2\pi i} \oint_C \frac{1}{(\zeta - z)^{m+1}} f(\zeta)\, d\zeta$$

Such an approach to the fractional calculus (of analytic functions) was explored by A. V. Litnikov, N. Ya. Sonin and H. Laurent in the 1870's and 1880's. For discussion, see Miller & Ross, p. 28 or Oldham & Spanier, p. 54.

The fractional calculus is, by this account, essentially a theory of *fractional integration,* within which fractional derivatives arise as secondary constructions:

$$ {}_aD_x^{m-\nu} f(x) = D^m \cdot {}_aD_x^{-\nu} f(x) $$

We would, for example, write

$$ \begin{aligned} {}_aD_x^{\frac{1}{3}} f(x) &= D^1 \cdot {}_aD_x^{-\frac{2}{3}} f(x) \\ {}_aD_x^{\frac{4}{3}} f(x) &= D^2 \cdot {}_aD_x^{-\frac{2}{3}} f(x) \end{aligned} $$

Look in particular to the "semiderivative"

$$ \begin{aligned} {}_aD_x^{\frac{1}{2}} f(x) &= D^1 \cdot {}_aD_x^{-\frac{1}{2}} f(x) \\ &= \frac{d}{dx} \cdot \frac{1}{\Gamma(\frac{1}{2})} \int_a^x \frac{1}{\sqrt{x-y}} f(y)\, dy \end{aligned} $$

which in the simple case $f(x) \equiv 1$ gives

$$ \begin{aligned} {}_aD_x^{\frac{1}{2}} 1 &= \frac{d}{dx} \cdot \frac{1}{\sqrt{\pi}} \int_a^x \frac{1}{\sqrt{x-y}}\, dy \\ &= \frac{d}{dx} \cdot \frac{2\sqrt{x-a}}{\sqrt{\pi}} \\ &= \frac{1}{\sqrt{\pi(x-a)}} \end{aligned} $$

and at $a = 0$ gives back Lacroix' result (10). Is this result surprising? Not, I think, when viewed in context: writing $D^{-\nu} \equiv {}_0D_x^{-\nu}$ and making free use of the identity $z\Gamma(z) = \Gamma(z+1)$, we obtain[12]

$$ \begin{aligned} D^{-\nu}1 \equiv u(x;\nu) &= \frac{1}{\Gamma(\nu)} \int_0^x (x-y)^{\nu-1}\, dy = \frac{1}{\nu\Gamma(\nu)} x^\nu \\ &= \frac{1}{\Gamma(1+\nu)} x^\nu \quad : \quad \nu > 0 \\ D^{+\mu}1 \equiv D^1 \cdot D^{-(1-\mu)}1 &= D \cdot u(x; 1-\mu) = \frac{(1-\mu)}{\Gamma(2-\mu)} x^{-\mu} \\ &= \frac{1}{\Gamma(1-\mu)} x^{-\mu} \quad : \quad 0 < \mu < 1 \end{aligned} $$

Curiously, this variant of the same argument

$$ \begin{aligned} D^{+\mu}1 = D^2 \cdot D^{-(2-\mu)}1 &= D^2 \cdot u(x; 2-\mu) = \frac{(2-\mu)(1-\mu)}{\Gamma(3-\mu)} x^{-\mu} \\ &= \frac{1}{\Gamma(1-\mu)} x^{-\mu} \quad : \quad 0 < \mu < 2 \end{aligned} $$

[12] All integrals here and henceforth have been supplied by *Mathematica*, unless otherwise noted. Note particularly the "weak divergence" $(0 < \nu < 1)$ of the integral here in question.

yields an identical result, but on an expanded domain. We conclude that it is possible in *all* cases ($\mu \lesseqgtr 0$) to write

$$D^\mu 1 \equiv U(x;\mu) = \frac{1}{\Gamma(1-\mu)}x^{-\mu} \tag{26}$$

and to attribute the familiar (but from this point of view remarkable) fact that $D^m 1 = 0$ ($m = 1, 2, 3, \ldots$) to the circumstance that $\Gamma(1-\mu)$ has poles at precisely those integral points.

Returning again to (25), we set $a = 0$ (this can always be achieved by a simple change of variable) and obtain

$$\begin{aligned} D^{-\nu} f(x) &= \frac{1}{\Gamma(\nu)}\int_0^x (x-y)^{\nu-1} f(y)\,dy \quad : \quad \nu > 0 \\ &\equiv \text{Riemann-Liouville “fractional integral transform” of } f(x) \end{aligned} \tag{27}$$

The fractional integral transform is somewhat reminiscent of the

$$\text{Hilbert transform}[f(x)] \equiv \tfrac{1}{\pi}\cdot \ \mathsf{P}\!\!\int_{-\infty}^{+\infty} (x-y)^{-1} f(y)\,dy$$

A good table of fractional integral transforms can be found in Chapter XIII of A. Erdelyi *et al*, *Tables of Integral Transforms; Bateman Manuscript Project* (1954).

4. Further elaborations. Within the ordinary calculus one builds upon such primitive statements as

- LINEARITY: $D(f+g) = Df + Dg$
- PRODUCT RULE: $D(f\cdot g) = Df\cdot g + f\cdot Dg$
- CHAIN RULE: $Df(g(x)) = \frac{df}{dg}\cdot\frac{dg}{dx}$

to develop an arsenal of general computational formulæ and procedures, of which

- LEIBNIZ' FORMULA: $D^m(f\cdot g) = \displaystyle\sum_{k=0}^{m}\binom{m}{k} D^{m-k}f\cdot D^k g$
- INTEGRATION BY PARTS: $(f\cdot g) = \int Df\cdot g + \int f\cdot Dg$
- SCALING LAW: $D^m f(\lambda x) = \lambda^m f^{(m)}(\lambda x)$

are typical. In a more comprehensive account of the fractional calculus one would want, at about this point, to construct fractional generalizations of those formulæ. This is in fact done in the standard monographs; the details are found to depend markedly upon whether one proceeds within the bounds of the Riemann-Liouville formalism, the Grünwald formalism or some variant of those, but the results obtained are (generally speaking) consonant. Here I must be content to remark simply that finite sums encountered in formulæ of the ordinary calculus tend generally within the fractional calculus to become infinite sums; Leibniz' formula, for example, becomes (in one of its variant forms)

$$D^\mu(f\cdot g)=\sum_{k=0}^\infty \binom{\mu}{k} D^{\mu-k}f\cdot D^k g$$

I have already made several passing allusions to connections between the fractional calculus and the theory of the Laplace transform. Some people prefer, in fact, to consider the former subject to be specialized sub-topic within the latter; it is, in any event, certainly the case that Laplace transform theory has from the beginning—tacitly, if not always explicitly—played a major role in the development and application of the fractional calculus. I record here only a few general observations in that bear on that aspect of our subject. If

$$\varphi(s)=\mathcal{L}[f(x)]\equiv\int_0^\infty e^{-sx}f(x)\,dx$$

then[13]

$$\begin{aligned}
(-)^n\varphi^{(n)}(s)&=\mathcal{L}[x^nf(x)]\\
s^n\varphi(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)&=\mathcal{L}[f^{(n)}(x)]\\
\int_s^\infty\int_{s_n}^\infty\cdots\int_{s_2}^\infty\varphi(s_1)\,ds_1ds_2\cdots ds_n&=\mathcal{L}[x^{-n}f(x)]\\
s^{-n}\varphi(s)=\mathcal{L}[\int_0^x\int_0^{x_n}\cdots\int_0^{x_2}f(x_1)\,dx_1dx_2\cdots dx_n]&
\end{aligned}$$

which are in some respects strongly reminiscent of (7). The expressions on the right/left/right/left sides of the preceding equations invite relaxation of any presumption that n be an integer. Special interest attaches to the observation that if $\varphi(s)=\mathcal{L}[f(x)]$ and $\gamma(s)=\mathcal{L}[g(x)]$ then

$$\varphi(s)\cdot\gamma(s)=\mathcal{L}[\int_0^x f(y)g(x-y)\,dy] \tag{28}$$

Since the right side of (28) involves precisely such a "convolution integral" as appears on the right side of (27), it would appear to be quite easy to discuss the Laplace transform properties of fractional integrals; specifically, we have

$$\begin{aligned}
\mathcal{L}[D^{-m}f(x)]&=\frac{1}{\Gamma(m)}\mathcal{L}[\int_0^x(x-y)^{m-1}f(y)\,dy]\\
&=\frac{1}{\Gamma(m)}\underbrace{\mathcal{L}[x^{m-1}]}_{=s^{-m}}\cdot\mathcal{L}[f(x)]
\end{aligned}$$

But Laplace transform methods, it should be borne in mind, are applicable only to a restricted subset of the set of functions susceptible to the more general methods of the factional calculus.[14]

[13] I take all the following statements from Chapter IV of A. Erdélyi *et al*, *Tables of Integral Transforms I* (1954).

[14] For further discussion of this topic, see Oldham & Spanier, §8.1 or Miller & Ross: Chapter III §6, Chapter IV §10 and Appendix C §4.

ILLUSTRATIVE APPLICATIONS

According to Miller & Ross, "the fractional calculus finds use in many fields of science and engineering, including fluid flow,[15] rheology, diffusive transport theory,[16] electrical networks, electromagnetic theory, probability and statistics ... viscoelasticity and"—of all subjects—"the electrochemistry of corrosion." And, though the vision of a fractional calculus came to Leibniz and others in moments of idle speculation, it does appear to the be case that concrete progress in the field was accomplished mainly by persons who drew their inspiration from specific problems of an applied nature. It is perhaps a measure only of my own limitations that I find most interesting the applications to mathematics itself, and to problems derived from physics. In following paragraphs I discuss two physical applications of historic (but continuing) interest, several derived from my own work, and one derived from the most recent issue of the journal *CHAOS*.

5. Abel's solution of the tautochrone problem. A mass m slides, under influence of gravity, along a frictionless wire, of which $x(y)$ serves to describe the figure. For convenience we place the bottom end of the wire at the origin: $x(0) = 0$. If the mass is released at height$h > 0$, then its speed $v(y)$ when is has descended to height y $(0 \le y \le h)$ is—by energy conservation, and irrespective of the figure of the wire—given by

$$v^2 = 2g(h - y)$$

Let $s(y)$ denote arc length, as measured along the wire from the origin to the point $[x(y), y] = [x(s), y(s)]$:

$$s(y) \equiv \int_0^y \sqrt{1 + \Big(\frac{dx}{dy}\Big)^2}\, dy$$

As the bead slides down the wire we have

$$v = -\frac{ds}{dt} = \underbrace{-s'(y)\dot{y}}_{\text{positive because } s'(y) > 0 \text{ and } \dot{y} < 0} = \sqrt{2g(h - y)}$$

giving

$$\tau(h) \equiv \text{time of descent} = \int_0^h \frac{1}{\sqrt{2g(h - y)}} s'(y)\, dy$$

---

[15] They discuss in enthusiastic detail its application to the design of a weir notch!

[16] See in this connection the final chapter in Oldham & Spanier.

The "tautochrone problem" asks for the design of the curve with the property that $\tau(h)$ is in fact independent of h.[17] To ask for such a curve $\mathcal{C}$ is, by notational adjustment of the result just obtained, to ask for the function $s(y)$ such that

$$\sqrt{2g}T = \int_0^y (y-z)^{-\frac{1}{2}} s'(z)dz \tag{29}$$

is constant. In writing (29) Abel launched (1823) what was to become the "theory of integral equations," and in his clever approach to the solution of (29) made the first practical application of what was to become the fractional calculus. Abel observed that (29) can—see again (25)—be written

$$\begin{aligned} D^{-\frac{1}{2}} s'(y) &= \sqrt{2g}\, T/\Gamma(\tfrac{1}{2}) \\ &= \sqrt{2gT^2/\pi} \equiv \mathcal{T}, \text{ a constant of prescribed value} \end{aligned}$$

and that therefore (by application of Lacroix' curious equation (10))

$$\begin{aligned} s'(y) &= D^{\frac{1}{2}}\mathcal{T} \\ &= \frac{\mathcal{T}}{\sqrt{\pi y}} = Ay^{-\frac{1}{2}} \quad \text{with } A \equiv \sqrt{2g\,(T/\pi)^2} \end{aligned} \tag{30}$$

By integration

$$s(y) = 2Ay^{\frac{1}{2}} \tag{31}$$

since $s(0) = 0$. Extraction of a cycloid from (31) entails only some relatively uninteresting analytical geometry; backing up to (30) we have

$$\sqrt{1 + \Big(\frac{dx}{dy}\Big)^2} = Ay^{-\frac{1}{2}}$$

giving

$$\begin{aligned} \frac{dx}{dy} &= \sqrt{A^2y^{-1} - 1} \\ x(y) &= \int_0^y \sqrt{A^2z^{-1}-1}\,dz \qquad &&\text{by } x(0) = 0 \\ &= 2A\int_0^{\beta\equiv\arcsin\sqrt{\frac{y}{A}}} \cos^2\varphi\, d\varphi &&\text{by } z \equiv A\sin^2\varphi \\ &= A(\beta + \tfrac{1}{2}\sin 2\beta) \end{aligned}$$

[17] The tautochrone problem is not to be confused with the "brachistochrone problem," which had been discussed as early as 1630 by Galileo, was solved in 1696 by Johann Bernoulli (and, independently, by Newton and Leibniz), and asks for "the curve of quickest descent." Though the problems are distinct, they give rise to the same curve—the cycloid—so some confusion is almost inevitable. The brachistochrone problem served, as is well known, as a primary stimulus to the development of the calculus of variations.

Writing $\theta \equiv 2\beta$ and $R \equiv \frac{1}{2}A$ we therefore have

$$\begin{aligned} x(\theta) &= R(\theta + \sin\theta) && (32.1) \\ y(\theta) &= A\sin^2\tfrac{1}{2}\theta \\ &= R(1-\cos\theta) && (32.2) \end{aligned}$$

which—transparently—provides a parametric description of the cycloid traced when a circle of radius R rolls on the underside of the line $y = 2R$. Abel observed that integral equations of the generalized form

$$\int_0^y (y-z)^{-\alpha} s'(z)dz = \text{constant} \quad : \quad 0 < \alpha < 1$$

yield with equal ease to the line of argument which brought us to (31). We note (see below) that the success of that line of argument hinges critically on the linearity of the gravitational potential $U(y) = mgy$. And that the tautochrone problem had been posed and solved—by other means—long before Abel entered the picture; Huygens, by 1657, had discovered the tautochronous property of the cycloid and made it the basis of a famous horological invention.[18]

I digress here to pose this arcane question: *Does tautochronicity imply harmonicity?* The question derives from the familiar fact that if a mass m attached to a spring of strength k is removed from the origin to the point $x = A$ and then released, it returns to the origin in A-independent time

$$T = \tfrac{1}{4}(\text{period})$$

That's what me mean when we say that the oscillator is "harmonic." Evidently "harmonicity $\Rightarrow$ tautochronicity." To ask (as we now do) "Is $\Leftarrow$ also true?" is, in effect, to ask "For what potentials $U(x)$ does

$$T(x) \equiv \int_0^x \frac{1}{\sqrt{\frac{2}{m}[U(x) - U(y)]}}\,dy$$

have the property that $\frac{d}{dx}T(x) = 0?''$ The problem, thus formulated, appears on its face to lie—except in the " gravitational" case $U(x) = a + bx$ considered by Abel—beyond the purview of the fractional calculus, so I set it aside for another day; that the motion $s(t)$ of Abel's cycloidally constrained particle is

[18] It remains a mystery to me how Abel managed—at age twenty-one—to know so much about a subject that "hadn't been invented yet;" Liouville did not begin work in the field until 1832 (Abel had by then been dead for three years), and it appears to have been Abel's accomplishment that stimulated him to do so. Riemann was not even born until three years after Abel had published his work, and his own contribution to the field—"Versuch einer Auffassung der Integration und Differentiation," written in 1847 (by another twenty-one year old)—was published posthumously.

in fact harmonic is, however, quite easy to demonstrate: returning with (31) to the energy conservation equation that served as our point of departure, one has

$$\tfrac{1}{2}m\dot{s}^2 = E - \tfrac{1}{2}m\omega^2 s^2$$
$$\omega \equiv \frac{\pi}{2T} = \frac{2\pi}{\text{period}}$$

Evidently $s(t) = [2E/m\omega^2]^{\frac{1}{2}}\cos\omega t$, which establishes the point at issue.

6. Heaviside's solution of the diffusion equation. The one-dimensional heat equation (diffusion equation) reads

$$\Big\{a\Big(\frac{\partial}{\partial x}\Big)^2 - \frac{\partial}{\partial t}\Big\}\psi(x,t) = 0 \quad : \quad a > 0$$

and at $a = i(\hbar/2m)$ becomes the Schrödinger equation of a free particle. Many years ago I had occasion to develop[19] the following sequence of shamelessly formal manipulations: Write

$$\begin{gathered}\frac{\partial}{\partial t}\psi = aD^2\psi\\ \downarrow\\ \psi_t(x) = e^{atD^2}\psi_0(x)\end{gathered}$$

where $D \equiv \frac{\partial}{\partial x}$ has for the moment been treated as a constant. In consequence of the Gaussian integral formula

$$\int_{-\infty}^{+\infty} e^{-(ax^2+2bx+c)}\,dx = \sqrt{\frac{\pi}{a}}\exp\Big\{\frac{b^2-ac}{a}\Big\} \quad : \quad \Re(a) > 0$$

one has this integral representation of the operator e^{atD^2}:

$$e^{atD^2} = \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4at}\xi^2}e^{-\xi D}\,d\xi \quad : \quad \Re(1/4at) > 0 \tag{33}$$

Evidently

$$\begin{aligned}\psi_t(x) &= \frac{1}{\sqrt{4\pi at}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4at}\xi^2}\psi_0(x-\xi)\,d\xi \quad \text{by Taylor's theorem}\\ &= \int_{-\infty}^{+\infty} g(\xi,t)\psi_0(x-\xi)\,d\xi \qquad (34)\end{aligned}$$

$$g(x,t) \equiv \frac{1}{\sqrt{4\pi at}}e^{-\frac{1}{4at}x^2} \tag{35}$$

[19] See APPELL, GALILEAN & CONFORMAL TRANSFORMATIONS IN CLASSICAL/QUANTUM FREE PARTICLE DYNAMICS: RESEARCH NOTES 1976, p. 286.

One easily establishes that $g(x,t)$ is itself a solution of the heat equation, and has these special properties:

$$\int_{-\infty}^{+\infty} g(x,t)\,dx = 1$$
$$\lim_{t\downarrow 0} g(x,t) = \delta(x)$$

It is called by Widder[20] the "source solution." Clearly $g(-\xi,t) = g(\xi,t)$. A change of variables $\xi \longrightarrow y = x-\xi$ therefore brings (34) to the form

$$\begin{aligned}\psi_t(x) &= \int_{-\infty}^{+\infty} g(x-y,t)\psi_0(y)\,dy\\ &= \text{weighted superposition of } y\text{-centered source solutions}\end{aligned}$$

which shows $g(x-y,t)$ to be, in effect, the "Green's function" of the heat equation. Setting $a = i(\hbar/2m)$ we obtain

$$G(x-y,t) = \sqrt{\frac{m}{2\pi i\hbar t}}\exp\left\{\frac{i}{\hbar}\frac{m}{2}\frac{(x-y)^2}{t}\right\}$$

which is familiar quantum mechanically as the "free particle propagator"—produced here by the swiftest means known to me. Preceding manipulations do serve to illustrate the power of the "operator calculus," but have on their face nothing to do with the "fractional calculus." Suppose, however, we were to reverse our procedure, treating not $D \equiv \frac{\partial}{\partial x}$ but $p \equiv \frac{\partial}{\partial t}$ as an initial "constant." The heat equation, written

$$\frac{\partial^2}{\partial x^2}\psi_t = b^2 p\psi_t \quad\text{with}\quad b^2 \equiv \frac{1}{a} \tag{36}$$

then gives

$$\begin{aligned}\psi_t(x) &= Ae^{-bx\sqrt{p}} + Be^{+bx\sqrt{p}}\\ &\downarrow\\ &= Ae^{-bx\sqrt{p}} \quad\text{if, for convenience, we set } B=0\end{aligned}$$

We note in passing that to set $B = 0$ is, in effect, to stipulate that in place of (36) we will study this "factor" of the diffusion equation:

$$\frac{\partial}{\partial x}\psi_t = -b\sqrt{p}\,\psi_t \quad\text{with}\quad b>0 \tag{37}$$

[20] D. V. Widder, *The Heat Equation* (1975), p.10.

Writing

$$
\begin{aligned}
Ae^{-bx\sqrt{p}} &= \sum_{n=0}^{\infty}\frac{(-bx)^n}{n!}\Big(\frac{d}{dt}\Big)^{\frac{1}{2}n}A\\
&= A+\sum_{n\text{ odd}}^{\infty}\frac{(-bx)^n}{n!}\Big(\frac{d}{dt}\Big)^{\frac{1}{2}n}A\quad\text{by}\quad\Big(\frac{d}{dt}\Big)^{\text{non-zero integer}}A=0\\
&= A-\sum_{m=0}^{\infty}\frac{(bx)^{2m+1}}{(2m+1)!}\Big(\frac{d}{dt}\Big)^{m}\cdot\underbrace{\Big(\frac{d}{dt}\Big)^{\frac{1}{2}}A}_{=\frac{A}{\sqrt{\pi t}}\quad\text{by Lecroix' (10)}}\\
&= A\Big\{1-\tfrac{1}{\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{(-)^m}{m!}\underbrace{\frac{(bx)^{2m+1}}{(2m+1)2^{2m}t^{m+\frac{1}{2}}}}_{=2\int_0^{bx/2\sqrt{t}}\xi^{2m}\,d\xi}\Big\}
\end{aligned}
$$

we obtain this particular solution of

$$
\begin{aligned}
h(x,t) &= A\Big\{1-\tfrac{2}{\sqrt{\pi}}\int_0^{bx/2\sqrt{t}}e^{-\xi^2}d\xi\Big\}\\
&= A\Big\{1-\text{erf}(\tfrac{1}{2}bx/\sqrt{t})\Big\}\\
&= A\cdot\text{erfc}(\tfrac{1}{2}bx/\sqrt{t})
\end{aligned}
\tag{38}
$$

where I have appealed to the definitions of the "error funtion" $\text{erf}(x)$ and its complement $\text{erfc}(x)$. The preceding remarks have been adapted from Miller & Ross' somewhat disparaging account of a line of argument first advanced by Oliver Heaviside (1893) in connection with the theory of transmission lines, and have much in common—both in spirit and in detail—with §§8 & 9 of Widder's Chapter III . More recent work in this same ancient tradition has been concerned with the application of the fractional calculus to (for example) the study of the "fractional diffusion equation" and diffusion on fractal domains.[21]

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[21] See M. Giona & H. E. Roman, "Fractional diffusion equation on fractals: one-dimensional case & asymptotic behavior," J. Phys. A Math. Gen. **25**, 2093 (1992); H. E. Roman & M. Giona, "Fractional diffusion equation on fractals: three-dimensional case and scattering function," J. Phys. A Math. Gen. **25**, 2107 (1992). The "fractional diffusion equation" arises when, in place of (37), one writes

$$\frac{\partial}{\partial x}\psi_t = -bp^{1/\delta}\psi_t$$

where δ is the so-called "anomalous diffusion exponent;" in some applications (random walk on a Cantor set) δ turns out to be closely related to fractal dimension. A more recent reference (for which I am indebted to Oz Bonfim) is B. J. West *et al*, "Fractional diffusion and Lévy stable processes," Phys. Rev. E **55**, 99 (1997).

It is interesting to note that my "Gaussian method" and "Heaviside's method" led us to *distinct* solutions of the diffusion equation (which I find it convenient now to call the "heat equation"): the thermal Green's function $g(x,t)$ describes the temperature distribution that results when a brief localized pulse of heat is injected into an infinite rod, while $h(x,t)$ corresponds to the case in which one end of a semi-infinite rod is (by continuous heat injection) maintained at a constant temperature. The latter solution is quantum mechanically unfamiliar because it is inconsistent with probability conservation.

7. Riesz' "method of dimensional ascent." It was known already to d'Alembert (∼1770) that if $\varphi(x,0)$ and $\varphi_t(x,0)$ are the values assumed initially (i.e., at time $t=0$) by a field $\varphi(x,t)$ and its time-derivative, then the solution of the one-dimensional wave equation

$$\Box\varphi = 0 \quad : \quad \Box \equiv \partial_x^2 - \tfrac{1}{u^2}\partial_t^2 \text{ is the wave operator or "d'Alembertian"}$$

that evolves from that prescribed "Cauchy data" can be described[22]

$$\begin{aligned}\varphi(x,t) &= \tfrac{1}{2}\big\{\varphi(x+ut,0)+\varphi(x-ut,0)\big\} + \tfrac{1}{2u}\int_{x-ut}^{x+ut}\varphi_t(y,0)\,dy \qquad (39)\\ &= [\text{average of contributing } \varphi(x,0)\text{-values}]\\ &\qquad\qquad + [\text{average of contributing } \varphi_t(x,0)\text{-values}]\end{aligned}$$

d'Alembert's formula (39) can be notated

$$\varphi(x,t) = \int_{-\infty}^{+\infty}\Big\{\varphi(y,0)G_t(x-y,t)+\varphi_t(y,0)G(x-y,t)\Big\}\,dy \qquad (40)$$

[22] The constructive argument runs as follows: write (as one invariably can)

$$\varphi(x,t) = f(x+ut)+g(x-ut)$$

Then

$$\begin{aligned}\varphi(x,0) &= f\,(x)+g\,(x)\\ \tfrac{1}{u}\varphi_t(x,0) &= f'(x)-g'(x)\\ &\Downarrow\\ \tfrac{1}{u}\int_{-\infty}^{x}\varphi_t(y,0)\,dy &= f\,(x)-g\,(x)\end{aligned}$$

give

$$\begin{aligned}f(x) &= \tfrac{1}{2}\Big[\varphi(x,0)+\tfrac{1}{u}\int_{-\infty}^{x}\varphi_t(y,0)\,dy\Big]\\ g(x) &= \tfrac{1}{2}\Big[\varphi(x,0)-\tfrac{1}{u}\int_{-\infty}^{x}\varphi_t(y,0)\,dy\Big]\end{aligned}$$

from which (39) follows at once. Or one can simply verify that (39) does in fact possess the stated properties.

where

$$\begin{aligned} G(x-y,t) &\equiv \tfrac{1}{2u}\big\{\theta\big(y-(x-ut)\big)-\theta\big(y-(x+ut)\big)\big\} \\ &= \tfrac{1}{2u}\big\{\theta(y-x+ut))-\theta(y-x-ut)\big\} \\ &= \pm\theta(u^2t^2-(y-x)^2) \quad \text{according as } t \gtrless 0 \\ &\Downarrow \\ G_t(x-y,t) &= \ \tfrac{1}{2}\big\{\delta(y-x+ut))+\delta(y-x-ut)\big\} \end{aligned} \tag{41}$$

The $\varphi(x,t)$ that appears on the left side of (40) satisfies the wave equation because $G(x-y,t)$ does (and so also, therefore, does $G_t(x-y,t)$); within the population of solutions, the particular solution $G(x-y,t)$ possesses these distinguishing/defining features:

$$\left.\begin{aligned} G\ (x-y,0) &= 0 \\ G_t(x-y,0) &= \delta(x-y) \end{aligned}\right\} \tag{42}$$

The function $G(x-y,t)$ describes the field that results when the quiescent field is given an initial "kick" at the point $x=y$, and (40) describes how general solutions $\varphi(x,t)$ are to be assembled by superposition of such special solutions. A wonderful feature of (40) is that it is structurally so robust; an argument of famous elegance[23] shows that the solutions of virtually any sensible wave equation can, in terms of prescribed Cauchy data, be described by an equation of type (40); all that changes, when one moves from wave system to wave system, is the precise meaning assigned to the "Green's function" $G(\boldsymbol{x}-\boldsymbol{y},t)$.

In the winter of 1981/82 I was motivated by Richard Crandall's then on-going experimental effort to "measure the mass of a photon"[24] to study the 3-dimensional wave system

$$(\Box_3+\mu^2)\varphi=0 \quad : \quad \Box_3 \equiv (\partial_x^2+\partial_y^2+\partial_z^2)-\tfrac{1}{u^2}\partial_t^2 \tag{43}$$

[23] For an account of that argument, see RELATIVISTIC CLASSICAL FIELDS (1973), p. 160 *et seq*. The argument hinges on a generalization of "Green's theorem"

$$\int_{\mathcal{R}}\big\{\phi\,\Box\psi-\psi\,\Box\phi\big\}\,d^nx=\int_{\partial\mathcal{R}}\big\{\phi\,\partial^\alpha\psi-\psi\,\partial^\alpha\phi\big\}\,d\sigma_\alpha$$

which was invented (1828) for essentially this purpose, and is itself usually considered to be a corollary of Stokes' theorem (which, however, came later).

[24] See R. E. Crandall, "Photon mass experiment," AJP **51**, 698 (1983); R. E. Crandall & N. A. Wheeler, "Klein-Gordon radio and the problem of photon mass," Il Nuovo Cimento **80B**, 231 (1984) and R. Leavitt, "A photon mass experiment: an experimental verification of Gauss's Law," (Reed College, 1983).

which gives back the 3-dimensional wave equation $\Box_3\varphi = 0$ in the limit $\mu \downarrow 0$.[25] I was, at the time, teaching the rudiments of elementary wave theory to a class of sophomores and savoring the splendid little monograph *The Mathematical Theory of Huygens' Principle* by B. B. Baker & E. T. Copson (2$^{\text{nd}}$ edition 1953), who were themselves strongly influenced by M. Riesz' then fairly recent success in clarifying (by appeal to a generalization of the fractional calculus) the work of J. Hadamard;[26] I found it therefore natural to study (43) in a context which considers dimension to be a variable, and to include the results of my research in my SOPHOMORE NOTES: INTRODUCTION TO THE ANALYTICAL METHODS OF PHYSICS (1981), where in all their extravagant detail they can be found on pp. 366–433. The wave systems treated there are the "free-field Klein-Gordon systems"

$$(\Box_N + \mu^2)\varphi = 0 \quad : \quad \Box_N \equiv \sum_{n=1}^{N} \partial_n^2 - \tfrac{1}{u^2}\partial_t \tag{44}$$

$$\Downarrow$$

$$\Box_N\varphi = 0 \quad \text{in the limiting case } \mu \downarrow 0$$

Standard Fourier transform techniques were found to lead[27] to (compare (40))

$$\varphi(\boldsymbol{x}, t) = \int\cdots\int_{-\infty}^{+\infty} \left\{ \varphi(\boldsymbol{y}, 0)\frac{\partial G_N(\boldsymbol{x}-\boldsymbol{y}, t)}{\partial t} + \varphi_t(\boldsymbol{y}, 0)G_N(\boldsymbol{x}-\boldsymbol{y}, t)\right\} d^N y \tag{45}$$

and

$$G_N(\boldsymbol{x}-\boldsymbol{y}, t) = \frac{1}{(2\pi)^N}\int\cdots\int_{-\infty}^{+\infty} \frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\, dk_1 dk_2\cdots dk_N \tag{46}$$

where $\boldsymbol{r} \equiv \boldsymbol{x} - \boldsymbol{y}$. The central analytical problem is—for reasons already recounted, and made explicitly evident by (45)—to describe the structure of the Green's functions $G_N(\boldsymbol{x}-\boldsymbol{y}, t)$. When one introduces "$\boldsymbol{r}$-adapted polar/spherical /hyperspherical coordinates" into $\boldsymbol{k}$-space it becomes possible (as it turns out) to carry out all the angular integrations, and to achieve

$$G_N(\boldsymbol{x}-\boldsymbol{y}, t) = \frac{1}{\sqrt{(2\pi)^N}}\int_0^{\infty} \frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}} \left\{ k^{N-1}\left[\frac{1}{kr}\right]^{\frac{N-2}{2}} J_{\frac{N-2}{2}}(kr)\right\} dk \tag{47}$$

where the fact that $G_N(\boldsymbol{x}-\boldsymbol{y}, t)$ depends upon its spatial arguments only *via* r serves to establish the rotational symmetry of the Green's function.[28] Looking

[25] In the projected physical application one sets $u \to c$, writes $\mu = mc/\hbar$ and interprets m to be the "mass of the photon."

[26] See *Lectures on Cauchy's Problem in Linear Differential Equations* (1923). Hadamard wrote under the influence principally of V. Volterra.

[27] It seems to me fairly remarkable that Fourier analytic methods lead almost *automatically* to (45), without explicit appeal to Green's theorem.

[28] I shall, in the light of this development, consider myself free henceforth to use the notations $G_N(\boldsymbol{x}-\boldsymbol{y}, t)$ and $G_N(r, t)$ interchangeably, as seems most appropriate to the matter at hand.

now in closer detail to the expression interior to the curly brackets (where all the r-dependence originates) we have

$$
\begin{aligned}
\Big\{\text{etc.}\Big\} &= \quad \frac{1}{(kr)^{-\frac{1}{2}}}J_{-\frac{1}{2}}(kr) && \text{when } N=1\\
&= \quad \frac{1}{(kr)^{0}}J_{0}\,(kr)\cdot k && \text{when } N=2\\
&= k^2\frac{1}{(kr)^{\frac{1}{2}}}J_{\frac{1}{2}}(kr) && \text{when } N=3\\
&= k^2\,\frac{1}{(kr)^{1}}J_{1}\,(kr)\cdot k && \text{when } N=4\\
&= k^4\frac{1}{(kr)^{\frac{3}{2}}}J_{\frac{3}{2}}(kr) && \text{when } N=5\\
&= k^4\,\frac{1}{(kr)^{2}}J_{2}\,(kr)\cdot k && \text{when } N=6\\
&\;\;\vdots
\end{aligned}
$$

—the general formula being (for $n=0,1,2,\ldots$)

$$
\Big\{\text{etc.}\Big\} = \begin{cases} k^{2n}\big[\frac{1}{kr}\big]^{n-\frac{1}{2}}J_{n-\frac{1}{2}}(kr) & \text{when } N=2n+1 \text{ is odd}\\ k^{2n}\big[\frac{1}{kr}\big]^{n}\quad J_{n}\quad (kr)\cdot k & \text{when } N=2n+2 \text{ is even}\end{cases}
$$

But[29]

$$
\frac{1}{z^{n+\nu}}J_{n+\nu}(z) = \Big(-\frac{1}{z}\frac{d}{dz}\Big)^n\Big\{\frac{1}{z^\nu}J_\nu(z)\Big\}
$$

and trivially $\big(\frac{1}{kr}\frac{d}{d(kr)}\big)^n = k^{-2n}\big(\frac{1}{r}\frac{\partial}{\partial r}\big)^n$, so we have

$$
\Big\{\text{etc.}\Big\} = \begin{cases} \big(-\frac{1}{r}\frac{\partial}{\partial r}\big)^n\underbrace{\sqrt{kr}J_{-\frac{1}{2}}(kr)}_{=\sqrt{\frac{2}{\pi}}\cos kr} & \text{when } N=2n+1 \text{ is odd}\\ \big(-\frac{1}{r}\frac{\partial}{\partial r}\big)^n J_0(kr)\cdot k & \text{when } N=2n+2 \text{ is even}\end{cases} \tag{48}
$$

Returning with this information to (47) we obtain

$$
G_{2n+1}(r,t) = \Big(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\Big)^n G_1(r,t) \tag{49.1}
$$

$$
G_{2n+2}(r,t) = \Big(-\frac{1}{2\pi r}\frac{\partial}{\partial r}\Big)^n G_2(r,t) \tag{49.2}
$$

[29] See G. N. Watson, *Theory of Bessel Functions* (1944), p. 46.

where

$$G_1(\boldsymbol{x}-\boldsymbol{y},t) \equiv G_1(r,t) = \frac{1}{\pi}\int_0^\infty \frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}}\cos kr\,dk \tag{50.1}$$

$$\begin{aligned} G_2(\boldsymbol{x}-\boldsymbol{y},t) \equiv G_2(r,t) &= \frac{1}{2\pi}\int_0^\infty \frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}}J_0(kr)k\,dk \\ &= \frac{1}{2\pi u\sqrt{r}}\int_0^\infty \sqrt{k}\,\frac{\sin(ut)\sqrt{\mu^2+k^2}}{\sqrt{\mu^2+k^2}}J_0(kr)\sqrt{kr}\,dk \end{aligned} \tag{50.2}$$

The Green's functions $G_1(r,t)$ and $G_2(r,t)$ acquire special importance from the circumstance that, according to (49), they are the "seeds" from which arise the parallel constructions

$$\left.\begin{aligned} &G_1(r,t)\to G_3(r,t)\to G_5(r,t)\to G_7(r,t)\to\cdots \\ &G_2(r,t)\to G_4(r,t)\to G_6(r,t)\to G_8(r,t)\to\cdots \end{aligned}\right\} \tag{51}$$

The integrals which at (50) serve to define $G_1(r,t)$ and $G_2(r,t)$ are tabulated; consulting Volume I of A. Erdélyi *et al*, *Tables of Integral Transforms* (1954) we find (at **1.7.30**, p.26 in the table of Fourier cosine transforms) that

$$\begin{aligned} G_1(r,t) &= \begin{cases} \pm\frac{1}{2u}J_0(\mu\sqrt{(ut)^2-r^2}) & \text{if}\quad (ut)^2-r^2\geq 0 \\ \\ 0 & \text{otherwise} \end{cases} \\ &= \pm\theta(s^2)\cdot\frac{1}{2u}J_0(\mu s) \qquad\qquad \text{with}\quad s\equiv\sqrt{(ut)^2-r^2} \end{aligned} \tag{52.1}$$

according as $t \gtrless 0$, while in Volume II of that same work (at **8.7.20**, p. 35 in the table of Hankel transforms[30]) we find that

$$\begin{aligned} G_2(r,t) &= \pm\frac{1}{2\pi u\sqrt{r}}\cdot\theta(s^2)\sqrt{\frac{\pi\mu r}{2s}}\,\underbrace{J_{-\frac{1}{2}}(\mu s)}_{=\sqrt{\frac{2}{\pi\mu s}}\cos\mu s} \\ &= \pm\theta(s^2)\cdot\frac{1}{2\pi u}\frac{\cos\mu s}{s} \end{aligned} \tag{52.2}$$

Equations (52) owe their analytical simplicity in part to the definition

$$s \equiv \sqrt{(ut)^2-r^2}$$

[30] By definition the Hankel transform of order ν sends

$$f(x) \longrightarrow \int_0^\infty f(x)J_\nu(xy)\sqrt{xy}\,dx \quad : \quad y>0$$

and become superficially less attractive when, in service of another kind of simplicity, we introduce

$$\sigma \equiv s^2 = (ut)^2 - r^2 \quad \text{which} \begin{cases} \text{vanishes on the lightcone} \\ \text{is positive interior to the lightcone} \\ \text{is negative exterior to the lightcone} \end{cases}$$

We take our motivation here from the observation that $\frac{\partial}{\partial r} = \frac{\partial \sigma}{\partial r}\frac{\partial}{\partial \sigma} = -2r\frac{\partial}{\partial \sigma}$ entails $-\frac{1}{2\pi r}\frac{\partial}{\partial r} = \frac{1}{\pi}\frac{\partial}{\partial \sigma}$ and permits (49) and (51) to be notated

$$\mathsf{G}_{2n+1}(\sigma) = \left(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\right)^n \mathsf{G}_1(\sigma) \tag{53.1}$$

$$\mathsf{G}_{2n+2}(\sigma) = \left(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\right)^n \mathsf{G}_2(\sigma) \tag{53.2}$$

and

$$\left.\begin{array}{l} \mathsf{G}_1(\sigma) \Rightarrow \mathsf{G}_3(\sigma) \Rightarrow \mathsf{G}_5(\sigma) \Rightarrow \mathsf{G}_7(\sigma) \Rightarrow \cdots \\ \mathsf{G}_2(\sigma) \Rightarrow \mathsf{G}_4(\sigma) \Rightarrow \mathsf{G}_6(\sigma) \Rightarrow \mathsf{G}_8(\sigma) \Rightarrow \cdots \end{array}\right\} \tag{54}$$

where $\mathsf{G}_N(\sigma(r,t)) = G_N(r,t)$ and where $\Rightarrow$ is accomplished by action of $\left(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\right)$. In this notation equations (52) read

$$\mathsf{G}_1(\sigma) = \pm\theta(\sigma)\cdot \frac{1}{2u}\, J_0(\mu\sqrt{\sigma}) \tag{55.1}$$

$$\mathsf{G}_2(\sigma) = \pm\theta(\sigma)\cdot \frac{1}{2\pi u}\frac{\cos\mu\sqrt{\sigma}}{\sqrt{\sigma}} \tag{55.2}$$

The Green's function $\mathsf{G}_N(\sigma)$ of the N-dimensional Klein-Gordon equation goes over into the Green's function of the associated wave equation in the limit $\mu \downarrow 0$. Writing

$$\mathsf{G}^0_N(\sigma) \equiv \lim_{\mu\downarrow 0} \mathsf{G}_N(\sigma)$$

we make use of the fact that $J_0(0) = 1$ to achieve

$$\mathsf{G}^0_1(\sigma) = \pm\theta(\sigma)\cdot \frac{1}{2u} \tag{56.1}$$

$$\mathsf{G}^0_2(\sigma) = \pm\theta(\sigma)\cdot \frac{1}{2\pi u}\frac{1}{\sqrt{\sigma}} \tag{56.2}$$

which are remarkable for their simplicity. Equation (56.1) reproduces precisely (41), while by straightforward extension of a line of argument familiar from p. 16 we have

$$D^{\frac{1}{2}}\mathsf{G}^0_1(\sigma) = \pm\frac{1}{2u}D^1\cdot \underbrace{\frac{1}{\sqrt{\pi}}\int_0^\sigma (\sigma-\tau)^{-\frac{1}{2}}\theta(\tau)\,d\tau}_{= \begin{cases} 2\sqrt{\frac{\sigma}{\pi}} & \text{if} \quad \sigma > 0 \\ 0 & \text{otherwise} \end{cases}}$$

$$= \pm\theta(\sigma)\frac{1}{2u\sqrt{\pi\sigma}}$$

from which it follows—remarkably—that

$$\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{\frac{1}{2}}\mathsf{G}_1^0(\sigma)=\mathsf{G}_2^0(\sigma) \tag{57}$$

More generally, we can return to (55) with the corresponding extension of

$$\begin{aligned}D^{\frac{1}{2}}J_0(a\sqrt{x})&=D^1\cdot D^{-\frac{1}{2}}J_0(a\sqrt{x})\\&=D^1\cdot\frac{1}{\sqrt{\pi}}\int_0^x(x-y)^{-\frac{1}{2}}J_0(a\sqrt{y})\,dy\\&=D^1\cdot\frac{2\sin(a\sqrt{x})}{a\sqrt{\pi}}\quad\text{according to }\textit{Mathematica}\\&=\frac{\cos(a\sqrt{x})}{\sqrt{\pi x}}\end{aligned}$$

to obtain

$$\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{\frac{1}{2}}\mathsf{G}_1(\sigma)=\mathsf{G}_2(\sigma) \tag{58}$$

from which (57) can be recovered as a limiting case. Conversely,

$$\begin{aligned}\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{1}{2}}\mathsf{G}_2(\sigma)&=\pm\frac{1}{\sqrt{\pi}}\int_0^\sigma(\sigma-\tau)^{-\frac{1}{2}}\theta(\tau)\frac{1}{2\pi u}\frac{\cos\mu\sqrt{\tau}}{\sqrt{\tau}}\,d\tau\\&=\pm\theta(\sigma)\frac{\pi^{\frac{1}{2}}}{2\pi^{\frac{3}{2}}u}\cdot\underbrace{\int_0^\sigma(\sigma-\tau)^{-\frac{1}{2}}\frac{\cos\mu\sqrt{\tau}}{\sqrt{\tau}}\,d\tau}_{=\pi J_0(\mu\sqrt{\sigma})}\\&=\mathsf{G}_1(\sigma)\end{aligned}$$

so we have

$$\begin{aligned}\mathsf{G}_3(\sigma)&=\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{1}\mathsf{G}_1(\sigma)\\&=\underbrace{\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{1}\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{1}{2}}}_{\equiv\big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\big)^{+\frac{1}{2}}}\mathsf{G}_2(\sigma)\end{aligned}$$

The implication of the results now in hand is that (compare (53)) it makes sense to write

$$\mathsf{G}_N(\sigma)=\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{\frac{N-1}{2}}\mathsf{G}_1(\sigma) \tag{59}$$

and that in refinement of (54) we have

$$\mathsf{G}_1(\sigma)\Rightarrow\mathsf{G}_2(\sigma)\Rightarrow\mathsf{G}_3(\sigma)\Rightarrow\mathsf{G}_4(\sigma)\Rightarrow\mathsf{G}_5(\sigma)\Rightarrow\mathsf{G}_6(\sigma)\Rightarrow\cdots \tag{60}$$

according to which the Klein-Gordon Green's functions of all dimensional orders —whether even or odd—can be generated by repeated semidifferentiation of a single "seed." The scheme (60) provides weakly generalized expression of an idea developed by M. Riesz[31] in the 1930's. Riesz' idea is the relatively more useful complement of an idea developed in the 1920's by J. Hadamard, who gave the name "Method of Descent"[32] to implications of an observation that follows most transparently from (46):

$$\begin{aligned}
\int_{-\infty}^{+\infty} G_N(\boldsymbol{r},t)\,dr^N &= \int_{-\infty}^{+\infty}\Big\{\frac{1}{(2\pi)^N}\int\!\cdots\!\int_{-\infty}^{+\infty}\frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}}\\
&\qquad\cdot\exp\Big\{i\Big[\sum_{n=1}^{N-1}k_nr^n+k_Nr^N\Big]\Big\}\,dk_1dk_2\cdots dk_N\Big\}\,dr^N\\
&= \frac{1}{(2\pi)^{N-1}}\int\!\cdots\!\int_{-\infty}^{+\infty}\frac{\sin ut\sqrt{k^2+\mu^2}}{u\sqrt{k^2+\mu^2}}\underbrace{\Big\{\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{iKR}\,dR\Big\}}_{=\,\delta(K)}\\
&\qquad\cdot\exp\Big\{i\sum_{n=1}^{N-1}k_nr^n\Big\}\,dk_1dk_2\cdots dk_{N-1}dK\\
&= G_{N-1}(\boldsymbol{r},t)
\end{aligned}\tag{61}$$

Hadamard's result, reduced to its essentials, can be formulated

$$G_{N-1}(r,t)=\int_0^\infty G_N(\sqrt{r^2+s^2},t)\,ds$$

and can by (47) be considered to be an implication this special instance

$$\int_0^\infty \frac{J_\nu(k\sqrt{r^2+s^2})}{(r^2+s^2)^{\frac{1}{2}\nu}}\,ds=\sqrt{\frac{\pi}{2k}}\,\frac{J_{\nu-\frac{1}{2}}(kr)}{r^{\nu-\frac{1}{2}}}$$

of "Sonine's formula."[33] Hadamard's method achieves

$$\mathsf{G}_1(\sigma)\Leftarrow\mathsf{G}_2(\sigma)\Leftarrow\mathsf{G}_3(\sigma)\Leftarrow\mathsf{G}_4(\sigma)\Leftarrow\mathsf{G}_5(\sigma)\Leftarrow\mathsf{G}_6(\sigma)\Leftarrow\cdots\tag{62}$$

by integrating out successive degrees of freedom; that process is evidently—but non-obviously—equivalent to repeated application of the

$$\text{semiintegration operator}\equiv\Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{1}{2}}$$

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[31] See Chapter I §7 of B. B. Baker & E. T. Copson, *The Mathematical Theory of Huygens' Principle* (2^{nd} edition 1950).

[32] See p. 46 in the monograph just cited, and §§29, 70 & 164 in Hadamard's *Lectures on Cauchy's Problem in Linear Partial Differential Equations* (1923).

[33] See G. N. Watson, *A Treatise on the Theory of Bessel Functions* (1966), p. 417; W. Magnus & F. Oberhettinger, *Formulas & Theorems for the Functions of Mathematical Physics* (1954), p. 29.

Important physical conclusions follow swiftly and elegantly from Riesz' construction (60), which it becomes natural now to call the "Method of Ascent." To expose those I set $\mu = 0$ and retreat to the more explicit notation of (54). Looking first to the **odd-dimensional case**, we have

$$\begin{aligned}
\mathsf{G}_1^0(\sigma) &= \pm\frac{1}{2u}\theta(\sigma)\\
&\downarrow\\
\mathsf{G}_3^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^1\mathsf{G}_1^0(\sigma) = \pm\frac{1}{2\pi u}\delta(\sigma)\\
\mathsf{G}_5^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^2\mathsf{G}_1^0(\sigma) = \pm\frac{1}{2\pi^2 u}\delta'(\sigma)\\
&\vdots\\
\mathsf{G}_{2n+1}^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^n\mathsf{G}_1^0(\sigma) = \pm\frac{1}{2\pi^n u}\delta^{(n)}(\sigma)
\end{aligned}\tag{63.1}$$

In the **even-dimensional case** the situation is significantly more complicated and qualitatively distinct, but for the simplest of reasons; we find

$$\begin{aligned}
\mathsf{G}_2^0(\sigma) &= \pm\frac{1}{2\pi u}\sigma^{-\frac{1}{2}}\theta(\sigma)\\
&\downarrow\\
\mathsf{G}_4^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^1\mathsf{G}_2^0(\sigma) = \pm\frac{1}{2\pi^2 u}\Big\{\sigma^{-\frac{1}{2}}\delta(\sigma) - \tfrac{1}{2}\sigma^{-\frac{3}{2}}\theta(\sigma)\Big\}\\
\mathsf{G}_6^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^2\mathsf{G}_2^0(\sigma) = \pm\frac{1}{2\pi^3 u}\Big\{\sigma^{-\frac{1}{2}}\delta'(\sigma) - \sigma^{-\frac{3}{2}}\delta(\sigma) + \tfrac{3}{4}\sigma^{-\frac{5}{2}}\theta(\sigma)\Big\}\\
&\vdots\\
\mathsf{G}_{2n+2}^0(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^n\mathsf{G}_2^0(\sigma) = \pm\frac{1}{2\pi^{n+1} u}\sum_{p=0}^{n}\binom{n}{p}\big[\sigma^{-\frac{1}{2}}\big]^{(p)}\big[\theta(\sigma)\big]^{(n-p)}
\end{aligned}\tag{63.2}$$

where $\big[\sigma^{-\frac{1}{2}}\big]^{(p)} = (-)^p\frac{(2p)!}{2^{2p}p!}\sigma^{-\frac{1}{2}-p}$ and $\big[\theta(\sigma)\big]^{(n-p)} = \delta^{(n-p-1)}(\sigma) : 0 \le p < n$. From (63) we learn that

- In all cases, $\mathsf{G}_N^0(\sigma)$ vanishes outside the lightcone;
- $\mathsf{G}_{\text{odd}\ge 3}^0(\sigma)$ is singular on the lightcone, but vanishes inside;
- $\mathsf{G}_{\text{even}}^0(\sigma)$ is singular on the lightcone, but—owing to the presence of a "dangling θ-function"—fails to vanish inside (as also does $\mathsf{G}_1^0(\sigma)$); it follows that radiative events in odd-dimensional spacetimes have persistent local effects. This is in sharp contrast to the situation in spacetimes of even dimension $N+1 \ge 4$.

Amongst the cases $\mathsf{G}_{\text{odd}\ge 3}^0(\sigma)$ the case $\mathsf{G}_3^0(\sigma)$—which refers, of course, to the world we physically inhabit—is special, in a sense which we are in position now to comprehend. Returning again to (47), we set $\mu = 0$ and after some simplification obtain

$$G^0_{\scriptscriptstyle N}(r,t) = \frac{1}{u}\Big(\frac{1}{2\pi}\Big)^{\frac{N}{2}}\sqrt{\frac{1}{r^{N-1}}}\int_0^\infty k^{\frac{N-3}{2}}\sin utk\Big\{J_{\frac{N-2}{2}}(kr)\sqrt{kr}\Big\}\,dk \tag{64}$$

As it happens, the functions $J_{\frac{2n-1}{2}}(z) : n = 0, 1, 2, \ldots$ (the so-called "spherical" Bessel Functions) are elementary; more specifically, we have

$$\begin{aligned}
J_{-\frac{1}{2}}(kr)\sqrt{kr} &= \sqrt{\frac{2}{\pi}}\cos kr\\
J_{+\frac{1}{2}}(kr)\sqrt{kr} &= \sqrt{\frac{2}{\pi}}\sin kr\\
J_{+\frac{3}{2}}(kr)\sqrt{kr} &= \sqrt{\frac{2}{\pi}}\Big[\frac{1}{kr}\sin kr - \cos kr\Big]\cos kr\\
&\vdots\\
J_{\frac{2n-1}{2}}(kr)\sqrt{kr} &= \sqrt{\frac{2}{\pi}}z^{\frac{2n-1}{2}}\Big(-\frac{1}{z}\frac{d}{dz}\Big)^{n-1}\frac{\sin z}{z}\Big|_{z=kr} \quad : \quad n = 1, 2, 3, \ldots
\end{aligned}$$

Returning with this information to (64) we have

$$\begin{aligned}
G^0_1(r,t) &= \frac{1}{\pi u}\int_0^\infty k^{-1}\underbrace{\sin utk\cdot\cos rk}_{=\frac{1}{2}\left[\sin k(r+ut) - \sin k(r-ut)\right]}\,dk\\
&= \text{weighted superposition of running waves}\\
&= \pm\theta(u^2t^2 - r^2)\cdot\frac{1}{2u}
\end{aligned}$$

where the final equation (supplied by *Mathematica*) reproduces precisely (56.1). Similarly

$$\begin{aligned}
G^0_3(r,t) &= \frac{1}{2\pi^2 ur}\int_0^\infty k^0\underbrace{\sin utk\cdot\sin rk}_{=\frac{1}{2}\left[-\cos k(r+ut) + \cos k(r-ut)\right]}\,dk\\
&= \textit{attenuated unweighted}\text{ superposition of running waves}\\
&= -\frac{1}{2\pi r}\frac{\partial}{\partial r}G^0_1(r,t)
\end{aligned}$$

which provides an explicit instance of (49.1). *Mathematica* reports here that the integral does not converge, but that is not cause for alarm; Green's functions are by nature distributions, intended to live always in the protective shade of an integral sign; convergence is achieved in applications by reversing the order of integration. Looking next to the case $N = 5$ (which is typical of the cases $N = 5, 7, 9, \ldots$) we have

$$\begin{aligned} G_5^0(r,t) &= \frac{1}{4\pi^3 u}\int_0^\infty \Big[\frac{1}{r^3}\sin utk\cdot\sin rk - \frac{k}{r^2}\sin utk\cdot\cos rk\Big]\,dk \\ &= \text{superposition of running waves with } \textit{distinct attenuation factors} \\ &= -\frac{1}{2\pi r}\frac{\partial}{\partial r}G_3^0(r,t) \\ &= \Big(\frac{1}{2\pi}\Big)^2 \frac{1}{r}\frac{\partial}{\partial r}\frac{1}{r}\frac{\partial}{\partial r}G_1^0(r,t) \end{aligned} \tag{65}$$

Wave propagation is, in this and higher-dimensional cases, *dispersive* because some $\frac{\partial}{\partial r}$-operators see $\frac{1}{r}$-factors standing to their right. The implication is that non-dispersive telegraphy is possible only

- in 2-dimensional spacetime (case $N = 1$), which is arguably too simple to support physicists, and
- in 4-dimensional spacetime (case $N = 3$), which manifestly is not

It is difficult to escape the feeling that the remarkable fact thus exposed must have something to do with "why space is 3-dimensional." The even-dimensional cases are non-contenders for reasons that we have traced to (10), i.e., to the curious fact that

$$\text{the semiderivative (with respect to } x\text{) of unity} = \frac{1}{\sqrt{\pi x}}$$

Remarks which are in many respects qualitatively similar pertain to the Klein-Gordon Green's functions $G_N(r,t)\ :\ \mu > 0$. All K-G systems are, however, dispersive, and the analytical details tend (as we have seen) to be more intricate.[34]

8. Application to the mensuration of hyperspheres. It is well known that the volume of an N-dimensional sphere of radius r is given by[35]

$$V_N(r) = \frac{\sqrt{\pi^N}}{\Gamma(1+\frac{N}{2})}r^N = \begin{cases} \frac{\pi^n}{p!}r^{2n} & \text{when } N = 2n \text{ is even} \\ 2\pi^n \frac{2^n}{1\cdot 3\cdot 5\cdots(2n+1)}r^{2n+1} & \text{when } N = 2n+1 \text{ is odd} \end{cases}$$

[34] After the preceding material—which is, as I have indicated, an abbreviated revision of material written in 1981/82—had been written out, I consulted Bob Reynolds' copy of Courant & Hilbert's *Methods of Mathematical Physics: Volume II* to discover whether "fractional calculus" is listed in the index. It is—at p. 523, in connection with a discussion of the application of the Heaviside calculus to the diffusion equation, and again at p. 702, in connection with a discussion of the wave equation. Both discussions (I am more interested than distressed to discover) run closely parallel to my own. . . and can, on those grounds, hardly be recommended too highly!

[35] Ref. Gradshteyn & Ryzhik, **4.632.2**, p.620. For derivation of the formula, see SOPHOMORE NOTES (1981), pp. 541–545.

On the other hand, it was established in §3 that fractional integration (with respect to a variable we elect now to call σ) of unity gives

$$D^{-\nu}1 \equiv \frac{1}{\Gamma(\nu)}\int_0^\sigma (\sigma - y)^{\nu-1}\,dy = \frac{1}{\Gamma(1+\nu)}\sigma^\nu$$

from which it follows in particular that

$$\begin{aligned}\sqrt{\pi^N}D^{-\frac{N}{2}}1 &= \frac{\sqrt{\pi^N}}{\Gamma(1+\frac{N}{2})}(\sqrt{\sigma})^N\\ &= V_N(\sqrt{\sigma})\\ &\equiv \mathsf{V}_N(\sigma)\end{aligned}$$

The implication is that to obtain $V_N(r)$ we have only to construct

$$\begin{aligned}\mathsf{V}_N(\sigma) &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{N}{2}}\mathsf{V}_0 \quad \text{with } \mathsf{V}_0 \equiv 1\\ &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{N-1}{2}}\mathsf{V}_1(\sigma)\\ &= \Big(\frac{1}{\pi}\frac{\partial}{\partial\sigma}\Big)^{-\frac{1}{2}}\mathsf{V}_{N-1}(\sigma)\end{aligned} \tag{66}$$

and then set $\sigma = r^2$. Equation (66) is—except for the *reversed sign in the exponent*—strongly reminiscent of (59), and gives rise to a " Riesz construction"

$$\mathsf{V}_1(\sigma) \Rightarrow \mathsf{V}_2(\sigma) \Rightarrow \mathsf{V}_3(\sigma) \Rightarrow \mathsf{V}_4(\sigma) \Rightarrow \mathsf{V}_5(\sigma) \Rightarrow \mathsf{V}_6(\sigma) \Rightarrow \cdots \tag{67}$$

that is strongly reminiscent of (60). The corresponding analog of Hadamard's "method of descent" now involves differentiation instead of integration.

My instincts tell me that something much deeper than mere analogy is going on here. I have cooked up a fancy way to say simple things about some simple geometrical objects. What I find exciting is the prospect, once the "analogy" is deeply understood, of saying equally simple things about the Green's functions of wave equations, and of thus avoiding altogether the complexities of §7.

Having several times stressed the importance of "escape from integrality," I digress to observe that

$$V_N(r) = \frac{\sqrt{\pi^N}}{\Gamma(1+\frac{N}{2})}r^N \quad \text{is meaningful even when } N \text{ is not an integer} \tag{68}$$

One has

$$\begin{aligned}
V_1(1) &= 2.00000 = 2\\
V_2(1) &= 3.14159 = \pi\\
V_3(1) &= 4.18879 = \tfrac{4}{3}\pi\\
V_4(1) &= 4.93480 = \tfrac{1}{2}\pi^2\\
V_5(1) &= 5.26379 = \tfrac{8}{15}\pi^2\\
V_6(1) &= 5.16771 = \tfrac{1}{6}\pi^3\\
V_7(1) &= 4.72477 = \tfrac{16}{105}\pi^3\\
V_8(1) &= 4.05871 = \tfrac{1}{24}\pi^4\\
V_9(1) &= 3.29851 = \tfrac{32}{945}\pi^4\\
V_{10}(1) &= 2.55016 = \tfrac{1}{120}\pi^5\\
&\vdots\\
V_\infty &= 0.00000
\end{aligned}$$

Looking to $\frac{d}{dN}\log V_{\scriptscriptstyle N}(1)$, we find that $V_{\scriptscriptstyle N}(1)$ is maximal when $\psi(\frac{N+2}{2}) = \log\pi$, where $\psi(z) \equiv \frac{d}{dz}\log\Gamma(z) = \Gamma'(z)/\Gamma(z)$ defines the "digamma function." With assistance from *Mathematica*, I discover that

$$V_{\scriptscriptstyle N}(1) \text{ is maximal in space of dimension } N = 5.2569464$$

9. Application to the differention of fractal curves. Karl Weierstrass certainly did not have fractals in mind, or the outreaches of 20^{th} Century physics, when he undertook to prove by counterexample that

"continuity does not imply differentiability"

This he did by exhibiting (at a meeting of the Berlin Academy in 1872) the curve that now bears his name. Related work had been done by Bernard Bolzano in 1834, by Bernard Riemann in the early 1860's and simultaneously by the Swiss mathematician Charles Cellérier—all of whom elected (as also did Weierstrass) not to publish their findings.[36]

For many years Weierstrass' creation lived in legend as a mathematical curiosity that physicists were happy not to have to worry about. But with

[36] For a sketch of the history of this subject, see B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (1977), p. 270. The function studied by Riemann has the form

$$R(t) = \sum_1^\infty n^{-2}\cos n^2 t$$

and turned out to be only "nearly but not quite nowhere differentiable;" see J.Gerver, "The differentiability of the Riemann function at certain rational multiples of π," Amer. J. of Math. **92**, 33 (1970).

recognition of the fractal aspects of the natural world all that has changed; one of the best recent accounts of the properties of the Weierstrass curve and its cognates is, in fact, the work of a pair of physicists, who study the function[37]

$$W(t) = \sum_{-\infty}^{\infty} \frac{[1 - e^{i\gamma^n t}]e^{i\varphi_n}}{\gamma^{(2-D)n}} \quad ; \quad 1 < D < 2;\ \gamma > 1;\ \varphi_n \text{ arbitrary}$$

which is found to be "continuous but non-differentiable" in the sense that (for all t) the W-series converges while the dW/dt-series does not. The functions $\Re[W(t)]$ and $\Im[W(t)]$ are reported to have Hausdorff-Besicovitch (or "fractal") dimension D. The function $W(t)$ becomes "deterministic" when one abandons the assumed randomness of φ_n, writing

$$\varphi_n = \mu n$$

The function then acquires the scaling property

$$W(\gamma t) = e^{-i\mu}\gamma^{2-D}W(t)$$

It serves to model one-dimensional Brownian motion at $D = 1.5$, and to model $1/f$ noise as $D \to 2$. Berry & Lewis provide many figures illustrative of the behavior (for assorted values of γ and D)of the functions

$$C(t) = +\Re[W(t)]\Big|_{\mu=0} = \sum_{-\infty}^{\infty} \frac{1 - \cos\gamma^n t}{\gamma^{(2-D)n}}$$

and

$$A(t) = -\Im[W(t)]\Big|_{\mu=\pi} = \sum_{-\infty}^{\infty} \frac{(-)^n \sin\gamma^n t}{\gamma^{(2-D)n}}$$

Berry ends, by the way, with the characteristically inspired out-of-the-hat observation that the one-dimensional quantum system

$$\begin{gathered}\psi'' + [E - U(x)]\psi = 0 \quad : \quad \psi(x) = 0 \text{ at } x = \pm\infty \\ U(x) = -A/x^2 \quad \text{with } A > \tfrac{1}{4}\end{gathered}$$

possesses an energy spectrum (note the infinitely deep ground state)

$$E_n = -\mathcal{E}_0 \cdot \gamma^n \quad : \quad n = 0, \pm 1, \pm 2, \ldots$$

that reproduces the "Weierstrass spectrum" of $W(t)$.

Curves evocative of the Weierstrass function are encountered when one looks to the Brownian motion which is presumed to underlie diffusive processes of various types. Dirac's "Zitterbevegung," though it has a weaker claim to

[37] M. Berry & Z. Lewis, "On the Weierstrass-Mandelbrot fractal function," Proc. Roy. Soc. London A **370**, 459 (1980).

literal physicality, inspires a similar train of thought,[38] as do typical realizations of Feynman's sum-over-paths formulation of quantum mechanics.[39]

The existence—and relevance to his fractal purposes—of (on the one hand) a fractional calculus[40] and (on the other) such a beast as the Weierstrass curve did not escape the notice of Mandelbrot, who appears to have been the first to make reference to both subjects in the same breath. Mandelbrot's interest in the fractional calculus apparently derives from his interest in the statistical properties of natural signals $f(t)$. He notes that integration is non-local, increases a function's smoothness, and that "smoothness equals local persistence"... but fractional integration (of order less than unity) has the opposite effect. That one "differentiates to expose and enhance variability" appears to have the status of a folk theorem, and provides the basis of a technique quite commonly used by experimentalists to extract signals from data[41] (even though, as John Essick has reminded me, "differentiation increases the noise"). The folk theorem seems hard to place on a secure mathematical base; certainly it does not pertain to functions of the form $f(t) = x^p$ or $f(t) = \sin \omega t$, but it does pertain, manifestly and familiarly, to $g(t) = e^{-at^2}$.

...Which brings me to the recent paper "Fractional differentiability of nowhere differentiable functions and dimensions" (CHAOS **6**, 505 (1996)) by K. M. Kolwankar & A. D. Gangal, which was brought to my attention yesterday by Oz Bonfim. The objective of these authors is to establish that $W(t)$, though non-differentiable, *is fractionally differentiable in a certain weakly specialized sense*, and that "maximal order of fractional differentiability" is simply related to the local scaling behavior ("box dimension"[42]) of $W(t)$. Kolwankar & Gangal explore the generalizability of their result, and argue that "local fractional derivatives provide a powerful tool for analysis of irregular and chaotic signals."

Kolwankar & Gangal take as their point of departure the equation

$$\begin{aligned} D^\mu f(x) &= D \cdot D^{-(1-\mu)} f(t) \\ &= D \cdot \frac{1}{\Gamma(1-\mu)} \int_a^x (x-y)^{-\mu} f(y)\, dy \quad : \quad 0 \le \mu < 1 \end{aligned}$$

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[38] See E. Merzbacher, *Quanatum Mecanics* (2^{nd} edition 1970), p. 598 or A. Massiah, *Quanatum Mecanics* (1970), p. 951.

[39] See, in this connection, Figure 7-1 in R. P. Feynman & A. R. Hibbs, *Quantum Mechanics and Path Integrals* (1965).

[40] The topic is mentioned at p. 298 in the monograph cited above, and at pp. 250 & 353 in *The Fractal Geometry of Nature* (1977).

[41] See, for example, Appendix A: "Modulation spectroscopy: the lock-in amplifier" in *The Art of Experimental Physics* (1991) by D. W. Preston & E. R. Dietz.

[42] For discussion of the relation of box dimension—sometimes called the "lower entropy index"—to Hausdorff-Besicovitch dimension see G. A. Edgar, *Measure, Topology, and Fractal Geometry* (1990), p. 185.

that serves within the Riemann-Liouville formalism to define the factional derivative

$$_aD_x^\mu f(x) \quad : \quad 0 \le \mu < 1$$

To evaluate what they call the "local fractional derivative" of $f(\xi)$ at $\xi = x$ they proceed as follows:

STEP ONE: Form the function $F(\xi) \equiv f(\xi) - f(x)$, which has by design the property that $F(x) = 0$. We note that if $f(\xi)$ is regular at $\xi = x$

$$f(\xi) = f(x) + \sum_{p=1}^\infty \frac{1}{p!} f^{(p)}(x)(\xi - x)^p$$

then

$$F(\xi) = \qquad\qquad \sum_{p=1}^\infty \frac{1}{p!} f^{(p)}(x)(\xi - x)^p$$

but that such a representation is precluded if x is a singular point; i.e., if $F(\xi)$ is, like the Weierstrass function, (at least locally) non-differentiable.

STEP TWO: Construct the ordinary (non-local) fractional derivative

$$_aD_x^\mu F(x) = D \cdot \frac{1}{\Gamma(1-\mu)} \int_a^x (x-y)^{-\mu} [f(y) - f(x)]\, dy$$

STEP THREE: Proceed to the limit (when it exists), writing

$$\begin{aligned} \mathsf{D}_x^\mu f(x) &\equiv \lim_{a \uparrow x} {}_aD_x^\mu F(x) \\ &= \frac{1}{\Gamma(1-\mu)} \cdot \lim_{a\uparrow x} D \int_a^x (x-y)^{-\mu} [f(y) - f(x)]\, dy \end{aligned} \tag{69}$$

where $D \equiv \frac{d}{dx}$ might more pedantically be notated D_x^1. Clearly

$$\mathsf{D}_x^\mu [f(x) + g(x)] = \mathsf{D}_x^\mu f(x) + \mathsf{D}_x^\mu g(x)$$

Also clear—but more interesting—is

$$\mathsf{D}_x^\mu\, C = 0 \quad : \quad C = \text{constant}$$

Therefore (and this was the intent of STEP ONE)

$$\mathsf{D}_x^\mu [f(x) + C] = \mathsf{D}_x^\mu f(x)$$

which is a familiar property of the differential operators D^{integer}, but a property not shared by the non-local operators $D^{-\text{integer}}$ and $_aD_x^\nu$. Equally elementary is the observation that

$$\begin{aligned} \mathsf{D}_x^0 f(x) &= \frac{1}{\Gamma(1)} \cdot \lim_{a\uparrow x} D \int_a^x (x-y)^{-0} [f(y) - f(x)]\, dy \\ &= \lim_{a\uparrow x} D \Big\{ \int_a^x f(y)\, dy - f(x)(x-a) \Big\} \\ &= \lim_{a\uparrow x} \Big\{ f(x) - f(x)(x-a) \Big\} \\ &= f(x) \end{aligned}$$

which makes attractive good sense; at $\mu = 0$ one has a kind of degenerate commonality of definition:

$$\mathcal{D}_x^0 f(x) = {}_aD_x^0 f(x) = D^0 f(x) = f(x) \quad : \quad \text{all } f(x)$$

Looking back again to the case $f(x) = C$, we have

$$\begin{aligned}
{}_aD_x^\nu\, C &= \frac{1}{\Gamma(1-\nu)\cdot(x-a)^\nu} C \quad : \quad 0 < a < x,\ \Re[\nu] < 1 \\
&= \text{Lacroix' curious equation (10) when } \nu = \tfrac{1}{2} \text{ and } a = 0 \\
&\neq \mathsf{D}_x^\nu\, C,\ \text{which was just seen to vanish for all } C
\end{aligned}$$

If x is a regular point of $f(x)$ (as it would, of course, be in the special case just studied: $f(x) = C$) then

$$\begin{aligned}
\mathcal{D}_x^\mu\, f(x) &= \lim_{a\uparrow x} D\cdot\frac{1}{\Gamma(1-\mu)}\int_a^x (x-y)^{-\mu}\sum_{p=1}^\infty \frac{1}{p!} f^{(p)}(x)(y-x)^p\,dy \\
&= \frac{1}{\Gamma(1-\mu)}\lim_{a\uparrow x} D\sum_{p=1}^\infty \frac{(-)^p}{p!} f^{(p)}(x)\int_a^x (x-y)^{p-\mu}\,dy \\
&= \frac{1}{\Gamma(1-\mu)}\lim_{a\uparrow x}\sum_{p=1}^\infty \frac{(-)^p}{p!} f^{(p)}(x)\cdot(p-\mu)\underbrace{\int_a^x (x-y)^{p-\mu-1}\,dy}_{=\frac{(x-a)^{p-\mu}}{p-\mu}} \\
&= \frac{1}{\Gamma(1-\mu)}\lim_{a\uparrow x}\sum_{p=1}^\infty \frac{(-)^p}{p!} f^{(p)}(x)\cdot(x-a)^{p-\mu} \\
&= 0
\end{aligned}$$

which gives back $\mathsf{D}_x^\mu\, f(x) = 0$ in the case $f(x) = C$.[43] Evidently

$$\mathsf{D}_x^\mu\, f(x) = 0 \quad \text{except at points where funny things are going on!} \tag{70}$$

Kolwankar & Gangal examine this simple variant

$$W_\lambda(t) \equiv \sum_{k=1}^\infty \frac{\sin\lambda^k t}{\lambda^{(2-D)k}}$$

of Weierstrass' $A(t)$ function, and manage with relative ease to establish that

$$\mathsf{D}_t^\mu\, W_\lambda(t) = 0 \quad \text{if and only if} \quad \mu < 2 - D \tag{71}$$

[43] The result just achieved is not to be confused (continued on the next page)

where D has by prior work been established to be the "box dimension" of $W_\lambda(t)$.[44] With somewhat greater effort they show more generally that if $f(t)$ is continuous then, under some fairly natural conditions, $\mathsf{D}_t^\mu\, f(t)$ exists, and is given by

$$\mathsf{D}_t^\mu\, f(t) = 0 \quad \text{if and only if} \quad \mu < \alpha \equiv 2 - D$$

where $D = \dim_B f(t)$ is the "box dimension" of $f(t)$ and α is its "critical order of fractional differentiability."

The "critical order" concept was recently shown to be useful in quite a different connection. In 1933 Paul Ehrenfest proposed a scheme for classifying thermodynamic phase transitions. One is to look to the derivatives $d^n F(\xi)/d\xi^n$ of certain thermodynamic potentials with respect to certain thermodynamic variables, and to say that $\xi = x$ marks the location of a "p^{th}-order phase transition" if p is the least value of n for which $d^n F(\xi)/d\xi^n|_{\xi=x}$ fails to exist. The function $F(\xi) = |\xi - x|^p$ provides the classic example. In a recent paper[45] R. Hilfer—proceeding along essentially the path first taken by Grünwald, but in evident ignorance of the fact that a "fractional calculus" exists—has been led to construction of the operator D_x^μ. His proposal is to construct $\mathsf{D}_x^\mu\, F(x)$ and, with μ now released from any requirement that it be an integrer, to associate $p = \mu_{\text{max}}$ with the "fractional Ehrenfest order" of the phase transition. He argues that such a procedure would serve to unify some aspects of the theory of phase transitions, would clear up certain long-standing puzzles, and would establish contact with the "multiscaling" concept as it is encountered in the theory of fractals.

(continued from the preceding page) with the statement (compare Oldham & Spanier, §4.4)

$$\begin{aligned}
{}_aD_x^\mu\, f(x) &= {}_aD_x^\mu \sum_{p=0}^\infty \frac{1}{p!} f^{(p)}(a)(x-a)^p \\
&= \sum_{p=0}^\infty \frac{1}{p!} f^{(p)}(a) \cdot \frac{1}{\Gamma(1-\mu)} D \int_a^x (x-y)^{-\mu}(y-a)^p\, dy \\
&= \sum_{p=0}^\infty \frac{1}{p!} f^{(p)}(a) \cdot \frac{(p-\mu+1)\Gamma(1-\mu)\Gamma(p+1)}{\Gamma(1-\mu)\Gamma(p-\mu+2)}\,(x-a)^{p-\mu} \\
&\qquad \text{Use } (p-\mu+1) = \frac{\Gamma(p-\mu+2)}{\Gamma(p-\mu+1)} \\
&= \sum_{p=0}^\infty \frac{1}{p!} f^{(p)}(a) \cdot \underbrace{\frac{\Gamma(p+1)}{\Gamma(p-\mu+1)}\,(x-a)^{p-\mu}}_{\text{Compare (31)}}
\end{aligned}$$

[44] See in this connection p. 277 of R. E. Crandall's *Projects in Scientific Computation* (1994).

[45] "Multiscaling and the classification of continuous phase transitions," Phys. Rev. Letters **68**, 190 (1992)

10. Charge density on a needle. Electric charge Q is is distributed with density $f(x)$ on the unit interval: $0 \le x \le 1$. Immediately

$$Q = \int_0^1 f(x)\,dx$$

while the electrostatic energy is (in suitable units) described by the functional

$$\begin{aligned} E[f(x)] &= \int_0^1\!\!\int_0^1 \frac{f(x)f(y)}{|x-y|}\,dydx \\ &= 2\int_0^1 f(x)\bigg\{\int_0^x \frac{1}{x-y}f(y)\,dy\bigg\}\,dx \end{aligned}$$

The expression interior to the brackets is in fact divergent, but only weakly so, in this sense:

$$\int_0^x \frac{1}{(x-y)^p}\,dy = \begin{cases} \infty & : \quad \Re[p] \ge 1 \\ \frac{x^{1-p}}{1-p} & : \quad \Re[p] < 1 \end{cases}$$

It sits, in other words, right at the "leading edge" of the divergent regime. Assuming the unit interval to be "conductive" (informally, a "needle"), we are led to consider this "tempered" version of a problem previously studied (with inconclusive results) by David Griffiths and his student, Ye Li:[46] Find the $f(x)$ which minimizes

$$F[f] \equiv \lim_{\nu\downarrow 0}\int_0^1 f(x)\bigg\{\int_0^x \frac{1}{(x-y)^{1-\nu}}f(y)\,dy\bigg\}\,dx \tag{72}$$

subject to the constraint

$$G[f] \equiv \int_0^1 f(x)\,dx = 1 \tag{73}$$

I look by way of preparation to a result achieved by Griffiths & Li, who by numerical analysis of two "bead models" of complementary design obtained data seemingly consistent with the *ansatz*

$$f(x) = A + \frac{B}{[x(1-x)]^{\frac{1}{3}}} \tag{74}$$

Mathematica supplies the information that

$$G[f] = A + gB \quad \text{with} \quad g \equiv \frac{\sqrt{\pi}\,\Gamma(\frac{2}{3})}{2^{\frac{1}{3}}\Gamma(\frac{7}{6})} = 2.053390$$

[46] "Charge density on a conducting needle," AJP **64**, 706 (1996).

Consistency with (73) entails that we set

$$B = k(1-A) \quad \text{with} \quad k \equiv 1/g = 0.4869995$$

We are led thus to the following one-parameter family of test functions

$$\begin{aligned} \hat{f}(x) \equiv A + B\phi(x) \quad \text{with} \quad \phi(x) &\equiv [x(1-x)]^{-\frac{1}{3}} \\ B = B(A) &\equiv 0.4869995(1-A) \end{aligned}$$

Returning with these to (72)—and suspending temporarily the operation $\nu \downarrow 0$ —we obtain

$$F[\hat{f}] = A^2 p + ABq + B^2 r$$

where $\{p, q, r\}$ are certain A-independent functions of ν which I will describe in a moment. Differentiation with respect to the adjustable parameter A gives

$$\begin{aligned} F'[\hat{f}] &= 2Ap + (B + AB')q + 2BB'r \quad \text{with} \quad B' = -k \\ &= 2Ap + k(1-2A)q - 2k^2(1-A)r \\ &= (2p - 2kq + 2k^2 r)A + (kq - 2k^2 r) \end{aligned}$$

To achieve $F'[\hat{f}] = 0$ we are obligated therefore to set

$$\begin{aligned} A &= \frac{(2k^2 r - kq)}{(2k^2 r - kq) + (2p - kq)} \\ &= \frac{1}{1+\alpha} \quad \text{with} \quad \alpha \equiv \frac{2p - kq}{2k^2 r - kq} \end{aligned} \tag{75}$$

Looking now to the construction of $p(\nu)$, $q(\nu)$ and $r(\nu)$ we find that the first of those functions

$$p(\nu) = \int_0^1 \left\{ \int_0^x \frac{1}{(x-y)^{1-\nu}}\, dy \right\} dx = \frac{1}{\nu(1+\nu)} \sim \frac{1}{\nu}$$

is easy to evaluate, but becomes singular in the limit. We infer on the basis of (75) that $q(\nu)$ and/or $r(\nu)$ must become similarly singular if a finite result is to be achieved in the limit. Looking now to those, we have

$$\begin{aligned} q(\nu) &= \underbrace{\int_0^1 \left\{ \int_0^x \frac{1}{(x-y)^{1-\nu}} \phi(y)\, dy \right\} dx}_{\equiv\, q_1(\nu)} + \underbrace{\int_0^1 \phi(x) \left\{ \int_0^x \frac{1}{(x-y)^{1-\nu}}\, dy \right\} dx}_{\equiv\, q_2(\nu)} \\ r(\nu) &= \int_0^1 \phi(x) \left\{ \int_0^x \frac{1}{(x-y)^{1-\nu}} \phi(y)\, dy \right\} dx \end{aligned}$$

and according to *Mathematica* it is asymptotically the case that

$$\begin{aligned} q_1(\nu) \text{ and } q_2(\nu) &\sim \frac{2.053390}{\nu} = \frac{1}{\nu} \cdot \frac{\Gamma^2(\frac{2}{3})}{\Gamma(\frac{3}{4})} \\ r(\nu) &\sim \frac{5.299916}{\nu} = \frac{1}{\nu} \cdot \frac{2^{\frac{7}{3}} \sqrt{3\pi}\, \Gamma(\frac{7}{6})}{\Gamma(\frac{2}{3})} \end{aligned}$$

We will find it convenient to write

$$q_1(\nu) + q_2(\nu) = q(\nu) = \tfrac{1}{\nu}\cdot Q \quad \text{with} \quad Q \equiv \Gamma^2(\tfrac{2}{3})/\Gamma(\tfrac{3}{4}) \qquad = 2.053390$$
$$r(\nu) = \tfrac{1}{\nu}\cdot R \quad \text{with} \quad R \equiv 2^{\frac{7}{3}}\sqrt{3\pi}\,\Gamma(\tfrac{7}{6})/\Gamma(\tfrac{2}{3}) = 5.299916$$

I remark in passing that the asymptotic statement $q_1(\nu) = q_2(\nu)$ is, though non-obvious, so striking that it should be susceptible to analytical proof, and might repay the effort.

The results now in hand put us in position to draw some fairly remarkable conclusions. It is evident that the functions $p(\nu)$, $q(\nu)$ and $r(\nu)$ are divergent in the limit $\nu \downarrow 0$, but *identically* divergent. Ratios of those functions are therefore well-defined in the limit. In particular, we by (75) have

$$\alpha = \frac{1-kQ}{k(kR-Q)} = 3.89145(1-kQ)$$

$$kQ = \frac{2^{\frac{1}{3}}\Gamma(\frac{7}{6})}{\sqrt{\pi}\,\Gamma(\frac{2}{3})}\cdot\frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{4}{3})} = 1.000000$$

Evidently $kQ = 1$ exactly, in conseqence of properties of the Γ-function which I will not linger to spell out. The implication is that $\alpha = 0$, therefore $A = 1$, therefore $B = 0$, therefore that—self-energy apart—the least energy achievable within the class of trial distributions $\hat{f}(x)$ is in fact achieved by the flat distribution

$$\hat{f}_{\min}(x) = 1 + \frac{0}{[x(1-x)]^{\frac{1}{3}}} \tag{76}$$

This result is at seeming variance from the result reported by Griffiths & Li,[47] but in recent conversation Griffiths has admitted to a growing suspicion that his figures are probably best read as an illustration of the *slowness of the approach* to an equilibrium distribution which he has come to suspect is in truth flat. In their paper, Griffiths & Li detect (in the behavior of models alternative to their bead models) certain ambiguous hints of a flatness which they "cannot absolutely exclude [as a] counterintuitive possibility," but seem more inclined to read those as evidence that "the [needle] problem is ill posed (in the sense that the answer depends upon the model adopted)."

The preceding discussion does serve to inspire confidence in the reliability of our regularization procedure $\nu \downarrow 0$, but cannot exclude the possibility that there might conceivably exist a non-flat distribution $f(x)$ with lower energy than the flat distribution, for we have actually established only that (76) is optimal *within one specific population* $\{\hat{f}(x)\}$ *of test functions*. To do better we must have recourse to more general methods. As a first step toward that objective, we recall the Riemann-Liouville definition (25) of the fractional integration operator

$$D^{-\nu}f(x) \equiv \frac{1}{\Gamma(\nu)}\int_0^x \frac{1}{(x-y)^{1-\nu}}f(y)\,dy \quad : \quad \nu > 0$$

[47] See especially their Figures 6, 7, 10 and 11.

and note that the energy functional (72) can in this notation be described

$$F[f] = \lim_{\nu\downarrow 0} \Gamma(\nu)\cdot\int_0^1 f(x)D^{-\nu}f(x)\,dx$$

I draw attention now to the fact that in (for example) the case

$$u(x)\equiv 1 \quad : \quad \text{unit flat distribution}$$

one has

$$\begin{aligned} F[u] &= \lim_{\nu\downarrow 0}\int_0^1\!\!\int_0^x \frac{1}{(x-y)^{1-\nu}}\,dy \\ &= \lim_{\nu\downarrow 0}\frac{1}{\nu(\nu+1)} \\ &= \infty \end{aligned}$$

I propose to resolve this instance of the "self-energy problem" not by subtraction of an infinite term, but by *dividing out* the factor responsible for the singularity. Noting that[48]

$$\frac{1}{\Gamma(\nu)} = \nu\Big\{1+\gamma\nu+\Big(\frac{\gamma^2}{2}+\frac{\pi^2}{12}\Big)+\cdots\Big\}$$

we construct the "renormalized energy functional"

$$\mathsf{F}[f] \equiv \frac{1}{\Gamma(\nu)}\cdot F[f] = \lim_{\nu\downarrow 0}\int_0^1 f(x)D^{-\nu}f(x)\,dx \tag{77}$$

Transparently, $\mathsf{F}[u]=1$: regularization has (at least in this typical case) been achieved, by a mechanism which which is "natural to the fractional calculus."[49]

It is in this modified context that we do our "calculus of variations." Writing

$$\begin{aligned} \mathsf{F}[f+\epsilon g] &= \mathsf{F}[f] + \epsilon\lim_{\nu\downarrow 0}\int_0^1 \big\{f(x)D^{-\nu}g(x)+g(x)D^{-\nu}f(x)\big\}\,dx+\cdots \\ G[f+\epsilon g] &= G[f]+\epsilon\int_0^1 g(x)\,dx \end{aligned}$$

we impose upon $f(x)$ the requirement that

$$\lim_{\nu\downarrow 0}\int_0^1 \big\{f(x)D^{-\nu}g(x)+g(x)D^{-\nu}f(x)\big\}\,dx = 0 \tag{78}$$

[48] Spanier & Oldnam, *Atlas of Functions* **43:6:1**, p. 415. Here $\gamma = 0.5772156$ is Euler's constant.

[49] Less naturally—but sufficiently for the purposes at hand—we could in place of the factor $1/\Gamma(\nu)$ introduce a factor of the form $\nu^{-1}\{1+a\nu+b\nu^2+\cdots\}$.

for all $g(x)$ such that

$$\int_0^1 g(x)\,dx = 0 \tag{79}$$

To reduce analytical clutter we shall, consistently with the physics of the problem, assume the functions $f + \epsilon g$ to be centrally symmetric, in the sense $f(x) = f(1-x)$. We can then (without loss of generality) write

$$\begin{aligned} f(x) &= f_0 + \sum_{n=1}^{\infty} f_n \cos 2\pi n x \\ g(x) &= g_0 + \sum_{n=1}^{\infty} g_n \cos 2\pi n x \\ g_0 &= 0 \quad \text{is required to insure compliance with (79)} \end{aligned}$$

Our problem now is to discover the conditions on $\{f_m\}$ which achieve

$$\frac{\delta\,\mathsf{F}[f]}{\delta g} = f_0 \sum_{n=1}^{\infty} W_{0n} g_n + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_m W_{mn} g_n = 0 \quad : \quad \text{all } \{g_n\}$$

where

$$W_{mn} \equiv V_{mn} + V_{nm}$$

with

$$\begin{aligned} V_{mn} &\equiv \lim_{\nu\downarrow 0} \int_0^1 \big\{ \cos 2\pi m x \cdot D^{-\nu} \cos 2\pi n x \big\}\,dx \\ &= \lim_{\nu\downarrow 0} \int_0^1 \cos 2\pi m x \bigg\{ \frac{1}{\Gamma(\nu)} \int_0^x \frac{1}{(x-y)^{1-\nu}} \cos 2\pi n y\,dy \bigg\}\,dx \end{aligned}$$

Now (in the notation of *Mathematica*, who did the work)

$$\begin{aligned} \frac{1}{\Gamma(\nu)} \int_0^x \frac{1}{(x-y)^{1-\nu}} \cos 2\pi n y\,dy &= \tfrac{x^\nu}{\nu\Gamma(\nu)} \cdot {}_pF_q\big(\{1\}, \{\tfrac{1}{2} + \tfrac{\nu}{2}, 1 + \tfrac{\nu}{2}\}, -n^2\pi^2x^2\big) \\ &\downarrow \\ &= \cos 2\pi n x \quad \text{in the limit } \nu \downarrow 0 \end{aligned}$$

so

$$\begin{aligned} V_{0n} &= \int_0^1 \cos 2\pi n x\,dx = 0 &&: \quad n = 1, 2, 3, \ldots \\ V_{mn} &= \int_0^1 \cos 2\pi n x \cos 2\pi m x\,dx = \tfrac{1}{2}\delta_{mn} &&: \quad m, n = 1, 2, 3, \ldots \end{aligned}$$

From this information it follows that

$$\begin{aligned} \frac{\delta\,\mathsf{F}[f]}{\delta g} &= \sum_{m=1}^{\infty} f_n g_n \\ &= 0 \quad \text{for all } \{g_n\} \text{ if an only if } f_n = 0\text{: } n = 1, 2, 3, \ldots \end{aligned}$$

We conclude that the functional $\mathsf{F}[f]$ is minimized at the flat distribution

$$f(x) = f_0$$

and (trivially) that normalization $\mathsf{F}[f] = 1$ entails $f_0 = 1$. Remarkably, we have achieved this result even though the energy functional (self-energy effects uncompensated) has infinite value:

$$E[f_{\text{flat}}(x)] = \infty$$

It is gratifying, in a way, that our result is consistent with the seeming analytical implications of the "ellipsoidal" and "cylindrical" models developed by Griffiths and Li. The interesting question appears to be this: "Why do bead models converge so slowly?" But to pursue that question would (or would it?) be to abandon our subject of the moment, which is the fractional calculus.

11. Eigenfunctions of derivative operators of integral/fractional order. In §6 we encountered the "fractional diffusion equation," which is a partial differential equation of fractional order. Prior to serious entry into such a subject one would want to become conversant with the theory of ordinary differential equations of fractional order, and the simplest considerations suggest that such a theory will embody some novel features. For example; we know that the general solution of the first of the following equations contains one adjustable constant (typically taken to be x_0), and that the general solution of

$$\begin{aligned}
\left(\tfrac{d}{dt}\right)^1 x(t) &= 0\\
\left(\tfrac{d}{dt}\right)^2 x(t) &= 0\\
\left(\tfrac{d}{dt}\right)^{\frac{1}{2}} x(t) &= 0
\end{aligned}$$

the second equation contains two adjustable constants (typically taken to be x_0 and $\dot{x}_0$). But how many adjustable constants enter into the general solution of the third equation?

In the ordinary theory of ordinary differential equations one draws at every turn upon the familiar fact that

$$D^n e^x = e^x \quad : \quad n = 0, 1, 2, \ldots$$

It is therefore interesting to note that (recall (25))

$$\begin{aligned}
D^{-\frac{1}{2}} e^x &\equiv \frac{1}{\Gamma(\frac{1}{2})} \int_0^x \frac{1}{\sqrt{x-y}}\, e^y\, dy\\
&= e^x \text{erf}(\sqrt{x}\,)\\
DD^{-\frac{1}{2}} e^x \equiv D^{+\frac{1}{2}} e^x &= e^x \text{erf}(\sqrt{x}\,) + \tfrac{1}{\sqrt{\pi x}}\\
D^{+\frac{3}{2}} e^x &= e^x \text{erf}(\sqrt{x}\,) + \tfrac{1}{\sqrt{\pi x}}\left\{1 - \tfrac{1}{2x}\right\}\\
D^{+\frac{5}{2}} e^x &= e^x \text{erf}(\sqrt{x}\,) + \tfrac{1}{\sqrt{\pi x}}\left\{1 - \tfrac{1}{2x} + \tfrac{3}{4x^2}\right\}\\
&\ \ \vdots
\end{aligned} \tag{80}$$

More generally, one has

$$\begin{aligned}
D^{-\nu}e^{ax} &= \frac{1}{\Gamma(\nu)}\int_0^x (x-y)^{\nu-1}e^{ay}\,dy \quad : \quad \nu > 0\\
&= \frac{e^{ax}}{\Gamma(\nu)}\int_0^x t^{\nu-1}e^{-at}\,dt \quad \text{with } t \equiv x - y\\
&= a^{-\nu}e^{ax}\Big\{1 - \tfrac{\Gamma(\nu,ax)}{\Gamma(\nu)}\Big\} = x^{\nu}e^{ax}\gamma^*(\nu;ax)\\
&\equiv E(x;\nu,a)
\end{aligned}$$

where $\Gamma(\nu, ax)$ and $\gamma^*(\nu; ax)$ are variant forms of the "incomplete gamma function."[50] The importance of the (nameless) function $E(x;\nu,a)$ has been emphasized particularly by Miller & Ross, who list many of its properties in their Appendix C. Drawing upon some of that material, one has (for $n = 0, 1, 2, \ldots$)

$$\begin{aligned}
D^nD^{-\nu}e^{ax} = D^{n-\nu}e^{ax} &= D^nE(x;\nu,a)\\
&= E(x;\nu-n,a)
\end{aligned}$$

Since $D^{\nu}e^x \neq e^x$ unless n is a non-negative integer, the question arises: What function $e(x;\nu)$ does have the property that

$$D^{\nu}e(x;\nu) = e(x;\nu) \quad : \quad \nu > 0 \tag{81}$$

To approach the problem we look first in an unfamiliar way to the familiar case $\nu = 1$; we construct

$$e(x;1) = \big\{1 + D^{-1} + D^{-2} + D^{-3} + \cdots\big\}g(x;1)$$

and impose upon the "generator" $g(x;1)$ the requirement

$$Dg(x;1) = 0$$

so as to achieve

$$De(x;1) = \big\{0 + D^{-0} + D^{-1} + D^{-2} + \cdots\big\}g(x;1) = e(x;1)$$

If, in particular, we take $g(x;1) \equiv 1$ then we obtain

$$e(x;1) = 1 + x + \tfrac{1}{2!}x^2 + \tfrac{1}{3!}x^3 + \cdots = e^x$$

The same idea gives

$$\begin{aligned}
e(x;2) = \big\{1 + D^{-1} + D^{-2} + D^{-3} + \cdots\big\}&g(x;2)\\
D^2&g(x;2) = 0
\end{aligned}$$

[50] See, for example, Chapter 45 of Spanier & Oldham, *Atlas of Functions* or §**6.5** of Abramowitz & Stegun.

Necessarily $g(x;2) = g_0 + g_1 x$, so

$$\begin{aligned} e(x;2) &= g_0\big\{1 + \tfrac{1}{2!}x^2 + \tfrac{1}{4!}x^4 + \cdots\big\} + g_0\big\{x + \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots\big\} \\ &= g_0 \cosh x + g_1 \sinh x \end{aligned}$$

and we obviously have only to set $g_0 = g_1 = 1$ to recover e^x. Proceeding similarly from $g(x;3) = g_0 + g_1 x + \frac{1}{2}g_2 x^2$ we obtain

$$e(x;3) = \sum_{k=0}^\infty \left\{ g_0 \frac{x^{3k}}{(3k)!} + g_1 \frac{x^{3k+1}}{(3k+1)!} + g_2 \frac{x^{3k+2}}{(3k+2)!} \right\}$$

More generally still, we have

$$e(x;n) = g_0 W_0(x;n) + g_1 W_1(x;n) + \cdots + g_{n-1} W_{n-1}(x;n) \tag{82}$$

where the functions

$$\begin{aligned} W_p(x;n) &\equiv \sum_{k=0}^\infty \frac{x^{kn+p}}{(kn+p)!} \quad : \quad p = 0,1,2,\ldots,n-1 & (83.1) \\ &= \sum_{k=0}^\infty \frac{1}{\Gamma(1+kn+p)} x^{kn+p} & (83.2) \end{aligned}$$

are individually solutions of

$$D^n f(x) = f(x) \quad : \quad n = 0,1,2,\ldots \tag{84}$$

and collectively (by "interdigitation," as it were) give back the exponential:

$$\sum_{p=0}^{n-1} W_p(x;n) = e^x \tag{85}$$

Returning to the problem (81) that stimulated the preceding digression, we find it now entirely natural to construct

$$e(x;\nu) = \big\{1 + D^{-\nu} + D^{-2\nu} + D^{-3\nu} + \cdots\big\} g(x;\nu) \tag{86}$$

and to impose upon $g(x;\nu)$ the requirement that

$$D^\nu g(x;\nu) = 0 \tag{87}$$

This, by $D^\nu g \equiv D \cdot D^{\nu-1} g = D \cdot D^{-(1-\nu)} g$, entails

$$D^{-(1-\nu)} g(x) \equiv \frac{1}{\Gamma(1-\nu)} \int_0^x \frac{1}{(x-y)^\nu} g(y)\,dy = \text{constant}$$

This is a "soft" condition in the sense that it admits of a continuum of solutions, within which the particular solution

$$\begin{aligned} g(x;\nu) &= D^{1-\nu}u(x) \\ &\qquad u(x) = 1 \quad (\text{all } x) \\ &= \frac{1}{\Gamma(\nu)}x^{\nu-1} \quad \text{by Lacroix' construction (8)} \end{aligned}$$

is distinguished only by its exceptional simplicity.[51] Bringing

$$[D^{-\nu}]^k g(x;\nu) = D^{-k\nu}g(x;\nu) = \frac{1}{\Gamma((k+1)\nu)}x^{(k+1)\nu-1}$$

to (86) we obtain

$$\begin{aligned} \mathcal{E}(x;\nu) &\equiv \sum_{k=1}^{\infty} \frac{1}{\Gamma(k\nu)}x^{k\nu-1} \\ &= \frac{x^{\nu-1}}{\Gamma(\nu)} + \frac{x^{2\nu-1}}{\Gamma(2\nu)} + \frac{x^{3\nu-1}}{\Gamma(3\nu)} + \cdots \end{aligned} \tag{88}$$

From Lacroix' construction (8.1) it then follows that

$$\begin{aligned} D^\nu \mathcal{E}(x;\nu) &= \frac{\Gamma(\nu)}{\Gamma(0)}\frac{x^{0-1}}{\Gamma(\nu)} + \underbrace{\frac{\Gamma(2\nu)}{\Gamma(2\nu-\nu)}\frac{x^{2\nu-\nu-1}}{\Gamma(2\nu)} + \frac{\Gamma(3\nu)}{\Gamma(3\nu-\nu)}\frac{x^{3\nu-\nu-1}}{\Gamma(3\nu)} + \cdots}_{\mathcal{E}(x;\nu)} \\ &= \qquad 0 \qquad + \qquad \mathcal{E}(x;\nu) \end{aligned}$$

as required. The function $\mathcal{E}(x;\nu)$ is, as Miller & Ross have remarked, strongly reminiscent of a seldom-studied function[52]

$$E_\nu(z) \equiv \sum_{k=0}^{\infty} \frac{1}{\Gamma(1+k\nu)}z^k \tag{89}$$

introduced (in quite another connection) by G. Mittag-Leffler; an elementary argument shows, in fact, that

$$\mathcal{E}(x;\nu) = DE_\nu(x^\nu)$$

[51] And, especially in the case $\nu = \frac{1}{2}$ by its exceptional utility; we have shown—and will have need of the fact—that

$$D^{\frac{1}{2}}\tfrac{1}{\sqrt{\pi x}} = 0 \tag{90}$$

[52] See §6.13 of the nice monograph *Lectures on the Theory of Functions of a Complex Variable* by G. Sansone & J. Gerretsen (1960).

and it is an immediate implication of (83.2) that

$$W_0(x;n) = E_n(x^n)$$

Looking in particular cases (*Mathematica* appears to be unable to say anything useful in the general case) to the right side of (88), we find

$$\mathcal{E}(x;1) = e^x = W_0(x;1)$$

$$\begin{aligned} \mathcal{E}(x;2) &= \sinh x = W_1(x;2) \\ D\mathcal{E}(x;2) &= \cosh x = W_0(x;2) \end{aligned}$$

$$\begin{aligned} \mathcal{E}(x;3) &= \tfrac{1}{3}\Big\{e^x - \ \ e^{-\frac{1}{2}x}\cos\big(\tfrac{\sqrt{3}}{2}x\big) - \sqrt{3}e^{-\frac{1}{2}x}\sin\big(\tfrac{\sqrt{3}}{2}x\big)\Big\} = W_2(x;3) \\ D\mathcal{E}(x;3) &= \tfrac{1}{3}\Big\{e^x - \ \ e^{-\frac{1}{2}x}\cos\big(\tfrac{\sqrt{3}}{2}x\big) + \sqrt{3}e^{-\frac{1}{2}x}\sin\big(\tfrac{\sqrt{3}}{2}x\big)\Big\} = W_1(x;3) \\ D^2\mathcal{E}(x;3) &= \tfrac{1}{3}\Big\{e^x + 2e^{-\frac{1}{2}x}\cos\big(\tfrac{\sqrt{3}}{2}x\big)\Big\} \qquad\qquad\qquad\qquad = W_0(x;3) \end{aligned}$$

The situation is further clarified if (in the case $n = 2$) we write

$$\omega = \text{primitive square root of unity} = e^{i\pi} = -1$$

and observe that

$$\left.\begin{aligned} W_1(x;2) &= \tfrac{1}{2}(e^x + \omega\ e^{\omega x}) \\ W_0(x;2) = DW_1(x;2) &= \tfrac{1}{2}(e^x + \ \ \ e^{\omega x}) \end{aligned}\right\} \tag{91.1}$$

while if we take

$$\omega = \text{primitive cube root of unity} = e^{i\frac{2}{3}\pi} = -\tfrac{1}{2} + i\tfrac{\sqrt{3}}{2}$$

(which entails $\omega^2 = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$) we obtain

$$\left.\begin{aligned} W_2(x;3) &= \tfrac{1}{3}(e^x + \omega\ \ e^{\omega x} + \omega^2 e^{\omega^2 x}) \\ W_1(x;3) = D\ \ W_2(x;3) &= \tfrac{1}{3}(e^x + \omega^2 e^{\omega x} + \omega\ \ e^{\omega^2 x}) \\ W_0(x;3) = D^2 W_2(x;3) &= \tfrac{1}{3}(e^x + \ \ \ e^{\omega x} + \ \ \ e^{\omega^2 x}) \end{aligned}\right\} \tag{91.2}$$

from which (see again (85)) $W_0(x;3) + W_1(x;3) + W_2(x;3) = e^x$ follows now as a consequence of the pretty cyclotomic condition

$$1 + \omega + \omega^2 = 0 \tag{92}$$

Equations (91) generalize straightforwardly to higher integral values of n (no major difficulty attends the fact that (92) resolves into a *system* of equations when n is not prime), and provide an efficient starting point for many lines of argument. For example, it follows immediately from (91) by (92) that

$$W_p(0;n) = \begin{cases} 1 & \text{if } p = 0 \\ 0 & \text{if } p = 1, 2, \ldots, n-1 \end{cases}$$

Somewhat less trivially, we write

$$\begin{pmatrix} W_0(x;2) \\ W_1(x;2) \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & \omega \end{pmatrix}\begin{pmatrix} e^x \\ e^{\omega x} \end{pmatrix}$$

and obtain

$$\begin{aligned} W_0(x;2)W_0(y;2) + W_1(x;2)W_1(y;2) &= \tfrac{1}{4}\begin{pmatrix} e^x \\ e^{\omega x} \end{pmatrix}^{\mathsf{T}}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} e^y \\ e^{\omega y} \end{pmatrix} \\ &= \tfrac{1}{2}\big(e^{x+y} + e^{\omega(x+y)}\big) \\ &= W_0(x+y;2) \qquad (93) \\ &\downarrow \\ W_0(x;2)W_0(-x;2) + W_1(x;2)W_1(-x;2) &= 1 \end{aligned}$$

which are more familiar as the statements

$$\begin{aligned} \cosh x \cdot \cosh y + \sinh x \cdot \sinh y &= \cosh(x+y) \\ \cosh^2 x - \sinh^2 x &= 1 \end{aligned}$$

An variant of the same argument gives (note the unanticipated structure of the expression on the left, which will be seen to be forced)

$$\begin{aligned} &W_0(x;3)W_0(y;3) + W_1(x;3)W_2(y;3) + W_2(x;3)W_1(y;3) \\ &\quad = \tfrac{1}{9}\begin{pmatrix} e^x \\ e^{\omega x} \\ e^{\omega^2 x} \end{pmatrix}^{\mathsf{T}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}^{\mathsf{T}}\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}_{\mathbb{M}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\begin{pmatrix} e^x \\ e^{\omega x} \\ e^{\omega^2 x} \end{pmatrix} \\ &\quad = \tfrac{1}{9}\begin{pmatrix} e^x \\ e^{\omega x} \\ e^{\omega^2 x} \end{pmatrix}^{\mathsf{T}}\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} e^x \\ e^{\omega x} \\ e^{\omega^2 x} \end{pmatrix} \\ &\quad = \tfrac{1}{3}\big(e^{x+y} + e^{\omega(x+y)} + e^{\omega^2(x+y)}\big) \\ &\quad = W_0(x+y;3) \qquad (94) \\ &\quad \downarrow \\ &1 = W_0(x;3)W_0(-x;3) + W_1(x;3)W_2(-x;3) + W_2(x;3)W_1(-x;3) \end{aligned}$$

where the structure of $\mathbb{M}$ (curiously $\mathbb{M} \neq \mathbb{I}$) was discovered by momentary tinkering. These results indicate that the "identities" that in their familiar profusion attach to the circular and hyperbolic functions are but the tip of an unfamiliar iceburg. We stand, evidently, on the shore of an unexplored continent, where the air is heavy with the scent of latent group theory.

Looking next to the illustrative (but—for obscure reasons—uniquely tractable) fractional case $\nu = \frac{1}{2}$, we are informed by *Mathematica* (who refuses to sum the series, but can be tickled into reluctant cooperation) that

$$\begin{aligned} \mathcal{E}(x;\tfrac{1}{2}) &\equiv \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}k)} x^{\frac{1}{2}k-1} \\ &= e^x + \tfrac{1}{\sqrt{\pi x}}\Big\{1 + 2x + \tfrac{4}{3}x^2 + \tfrac{8}{15}x^3 + \tfrac{16}{105}x^4 + \tfrac{32}{945}x^5 + \tfrac{64}{10395}x^6 + \cdots\Big\} \\ &= e^x + \tfrac{1}{\sqrt{\pi x}} \sum_{k=0}^{\infty} \tfrac{k!}{(2k)!}(4x)^k \\ &= \tfrac{1}{\sqrt{\pi x}} + e^x \cdot \operatorname{erfc}(-\sqrt{x}\,) \end{aligned} \tag{95}$$

This is consistent with a result remarked by Oldham & Spanier, who at p. 122 achieve (95) not by calculation but simply by observing, in the course of other work, that[53]

$$\begin{aligned} D^{\frac{1}{2}} \tfrac{1}{\sqrt{\pi x}} &= 0 \\ D^{\frac{1}{2}} e^x \cdot \operatorname{erfc}(-\sqrt{x}\,) &= \tfrac{1}{\sqrt{\pi x}} + e^x \cdot \operatorname{erfc}(-\sqrt{x}\,) \end{aligned}$$

and that therefore

$$\begin{aligned} D^{\frac{1}{2}}\mathcal{E}(x;\tfrac{1}{2}) &= \big\{D^{\frac{1}{2}} \tfrac{1}{\sqrt{\pi x}}\big\} + \big\{D^{\frac{1}{2}} e^x \cdot \operatorname{erfc}(-\sqrt{x}\,)\big\} \\ &= \big\{\ \ 0\ \ \big\} + \big\{\tfrac{1}{\sqrt{\pi x}} + e^x \cdot \operatorname{erfc}(-\sqrt{x}\,)\big\} \\ &= \mathcal{E}(x;\tfrac{1}{2}) \end{aligned} \tag{93}$$

The function $\mathcal{E}(x;\frac{1}{2})$ possesses, according to (96), a property which is shared also by an infinitude of other functions

$$\begin{aligned} D\mathcal{E}(x;\tfrac{1}{2}) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}k-1)} x^{\frac{1}{2}k-2} \\ &= e^x \cdot \operatorname{erfc}(-\sqrt{x}\,) + \tfrac{1}{\sqrt{\pi x}}\Big\{1 - \tfrac{1}{2x}\Big\} \\ D^2\mathcal{E}(x;\tfrac{1}{2}) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}k-2)} x^{\frac{1}{2}k-3} \\ &= e^x \cdot \operatorname{erfc}(-\sqrt{x}\,) + \tfrac{1}{\sqrt{\pi x}}\Big\{1 - \tfrac{1}{2x} + \tfrac{3}{4x^2}\Big\} \\ &\ \vdots \\ D^p\mathcal{E}(x;\tfrac{1}{2}) &= \sum_{k=0}^{\infty} \frac{1}{\Gamma(\frac{1}{2}k-p)} x^{\frac{1}{2}k-(p+1)} \quad : \quad p = 0, 1, 2, \ldots \end{aligned}$$

[53] The first of the following equations reproduces (90), while the second is a corollary of (80)

These results suggest strongly that something very like the integral theory carries over into the non-integral domain. I have, however, had as yet no opportunity to try to develop analytical methods strong enough to permit systematic exploration of the details. A first objective, in such an effort, would be to develop a non-integral generalization of (91). As a first step in that direction I can report the following development:

I have remarked that *Mathematica* appears to have nothing useful to say concerning (88) in the general case. Material tabulated on p. 1023 of Abramowitz & Stegun led me to ask what *Mathematica* might have to say concerning the *Laplace transforms* of the functions $\mathcal{E}(x;\nu)$ and their cognates; I learn that

$$\mathcal{L}[\mathcal{E}(x;1)] = \frac{1}{s\ \ -1} = \mathcal{L}[W_0(x;1)]$$

$$\begin{aligned}\mathcal{L}[\mathcal{E}(x;2)] &= \frac{1}{s^2-1} = \mathcal{L}[W_1(x;2)]\\ \mathcal{L}[D\mathcal{E}(x;2)] &= \frac{s}{s^2-1} = \mathcal{L}[W_0(x;2)]\end{aligned}$$

$$\begin{aligned}\mathcal{L}[\mathcal{E}(x;3)] &= \frac{1}{s^3-1} = \mathcal{L}[W_2(x;3)]\\ \mathcal{L}[D\mathcal{E}(x;3)] &= \frac{s}{s^3-1} = \mathcal{L}[W_1(x;3)]\\ \mathcal{L}[D^2\mathcal{E}(x;3)] &= \frac{s^2}{s^3-1} = \mathcal{L}[W_0(x;3)]\end{aligned}$$

$$\mathcal{L}[\mathcal{E}(x;\tfrac{1}{2})] = \frac{1}{\sqrt{s}-1}$$

It becomes, in the light of these highly patterned results, natural to conjecture—which *Mathematica* promptly confirms—that in the general case

$$\begin{aligned}\mathcal{L}[\mathcal{E}(x;\nu)] &= \frac{1}{s^\nu-1}\\ \mathcal{L}[D\mathcal{E}(x;\nu)] &= \frac{s}{s^\nu-1}\\ &\ \vdots\end{aligned}$$

It is pretty to see the roots of unity emerge so naturally as *locations of the poles* of a function so closely associated with $\mathcal{E}(x;\nu)$. Curiously (or perhaps not), *Mathematica* is powerless to provide a description of $\mathcal{L}^{-1}[1/(s^\nu-1)]$.

In classical analysis one makes fairly heavy use of "shift rules" of various types, of which

$$(D+a)^n = e^{-ax}D^n e^{ax} \quad : \quad n = 0, \pm1, \pm2, \ldots$$

provide the familiar simplest instance. The fractional analogs of such rules will evidently be fairly intricate—partly for reasons developed above (replacement of e^{ax} by $\mathcal{E}(ax;\nu)$), and partly because of the relative complexity of the fractional generalization of Leibniz's formula.

Classical analysis is, of course, a subject of many parts, and provides a wide and diverse range of opportunities to test the resources of the fractional calculus. A systematic survey is out of the question; I conclude this discussion, therefore, with a couple of semi-random remarks:

Analysis recommends to our attention a population of functions describable by various instances of Rodrigues' formula[54]

$$f_n(x) = \frac{1}{w(x)}\Big(\frac{d}{dx}\Big)^n\big\{w(x)[g(x)]^n\big\}$$

It becomes natural—mathematically, and sometimes also physically—to inquire into the properties of the functions which result then the tacit presumption $n = 0, 1, 2, \ldots$ is relaxed. By way of illustration, recall that in the quantum theory of angular momentum one is led by an operator-algebraic line of argument[55] from

$$\begin{aligned}\mathsf{L}^2|\psi) &= \lambda|\psi)\\ \mathsf{L}_z|\psi) &= \mu|\psi)\end{aligned}$$

to the information that necessarily

$$\begin{aligned}&\lambda = \hbar^2\ell(\ell+1) \quad \text{with } \ell = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\\ &\mu = \hbar m \quad \text{with } m \in \{-\ell, -(\ell-1), \ldots, +(\ell-1), +\ell\}\end{aligned}$$

Looking to the particular representation $(\boldsymbol{x}|\psi)$ of $|\psi)$ of finds (in spherical coordinates) that necessarily

$$(\boldsymbol{x}|\psi) = F(r)\cdot P_\ell^m(cos\theta)e^{im\varphi}$$

and that it is the requirement that the wave function be single-valued that forces $|m|$—whence also ℓ—to be integer-valued. If one were to relax that requirement, one would acquire interest in (non-polynomial) Legendre functions of fractional order:

$$\begin{aligned}P_\nu(x) &\equiv \frac{1}{\Gamma(\nu+1)}D^\nu\left[\frac{x^2-1}{2}\right]^\nu\\ &= {}_2F_1(\nu+1, -\nu, 1; \tfrac{1}{2}(1-x))\end{aligned}$$

Such functions are discussed in good detail in Chapter 59 of Spanier & Oldham's *Atlas of Functions*.[56]

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[54] See Abramowitz & Stegun, Chapter 22: "Orthogonal Polynomials."

[55] See, for example, §4.3 of D. Griffiths, *Introduction to Quantum Mechanics* (1995).

[56] For remarks that pertain in more general terms to the present topic, see Miller & Ross, pp. 115 & 307.

At (33), in connection with my discussion of the diffusion equation, I had occasion to make use of a certain "integral representation trick." That trick admits of a great many variations, of which I here report one—a casual inspiration of the moment:

$$e^{aD} = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4a}\xi^2} e^{-\xi D^{\frac{1}{2}}}\,d\xi$$

which by Taylor's theorem entails

$$f(x+a) = \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4a}\xi^2} e^{-\xi D^{\frac{1}{2}}} f(x)\,d\xi$$

The expression on the right can be developed

$$\begin{aligned}
&= \frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty} e^{-\frac{1}{4a}\xi^2}\Big\{\sum_{n\text{ even}} - \sum_{n\text{ odd}}\Big\}\frac{\xi^n}{n!}D^{\frac{n}{2}}f(x)\,d\xi\\
&= \sum_{n\text{ even}}\Big\{\frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty}\frac{\xi^n}{n!}e^{-\frac{1}{4a}\xi^2}\,d\xi\Big\}D^{\frac{n}{2}}f(x)\\
&= \sum_{m=0}^{\infty}\Big\{\frac{1}{\sqrt{4\pi a}}\int_{-\infty}^{+\infty}\frac{\xi^{2m}}{(2m)!}e^{-\frac{1}{4a}\xi^2}\,d\xi\Big\}D^m f(x)\\
&= \sum_{m=0}^{\infty}\frac{a^m}{m!}D^m f(x)
\end{aligned}$$

Evidently

$$\int_{-\infty}^{+\infty}\xi^{2m}e^{-\frac{1}{4a}\xi^2}\,d\xi = \frac{(2m)!}{m!}\sqrt{4\pi a^{2m+1}} \quad : \quad m = 0,1,2,\ldots$$

This is a modest achievement, but at least it is correct. In functional analysis it is not uncommon[57] to find the equation

$$\mathsf{A}^{-\nu} = \frac{1}{\Gamma(\nu)}\int_0^{\infty} e^{-s\mathsf{A}}s^{\nu-1}\,ds$$

used to define fractional powers of quite general operators; the little argument just concluded is in that broad tradition.

Concluding remarks. The fractional calculus has a speculative history that stretches back for more than 300 years; the first significant application—marked even then by an elegance which has been typical—was published more than 150 years ago, and the modern foundations of the subject have beeen securely in place for well more than 100 years. Why, therefore, does the subject remain so

[57] See, for example, p. 158 of A. Friedman, *Partial Differential Equations* (1969).

relatively little known? Why do modern authors feel an almost invariable need to spell out the fundamentals before getting down to business, and why does an apologetic air attach so often to their work? Several factors, it seems to me, may be contributory:

There is, to my ear, something faintly off-putting about the very name of the field; "fractional calculus" sounds like a calculus that lives in the cracks (like atonal music), a subject one can plausibly expect to get along well enough without. The name broadcasts a signal seemingly calculated to engage the passionate interest of disestablishmentarian iconoclasts, and to rub more polite folk the wrong way. Nor is the term notable for its accuracy; when one writes D^μ one does not actually require that μ be rational, and one gives precise meaning to the operator by appeal to the *ordinary* calculus. How different might have been the recent history and present status of the subject had it been called what it is: the *theory of the Riemann-Liouville integral transform!* Such nomenclature is (rightly) evocative of nobel ancestory and workhorse utility. And it suggests to anyone familiar with the rich interconnectedness present within the theory of integral transforms that would in most contexts be counterproductive to force the right foot into the left shoe, to attempt to force the Riemann-Liouville transform to be what it (in general) isn't—a specialized attachment to the theory of the Laplace transform.

Of course, people tend to have most lively interest in problems accessible to the tools in their command, and to cultivate an interest in "exotic" tools only when the urgency of otherwise inaccessible new problems enforces that interest. Thus, in recent times, did tomography stimulate an interest in the pre-existing theory of the Radon transform. It is my intuition that the on-going explosion of interest in the fractal aspects of the world, in critical phenomena, in "physics in the near proximity of disaster"—those things, but by no means only those—will soon spark a more general interest in the fractional calculus as a tool of choice, and that some of the papers I have cited can be read as precursors of such a development.

The syllabi of the "Fractional Calculus 201" courses of the future will list a number of topics—some quite basic—I have had opportunity in these few pages to mention only glancingly or not at all. Many of those are developed in Oldham & Spanier and Miller & Ross, and in sources there cited; others appear to await development. There's opportunity aplenty within this field for anyone who feels there might reside a bit of Euler in his bones, or a bit of Kowalewski in hers.

Miller & Ross, in §8 of their Chapter I: Historical Survey, remark that "Some of the still-open questions are intriguing. For example: Is it possible to find a geometric interpretation for a fractional derivative of noninteger order?" The question is ancient—recall Leibniz' lament that the subject "seems removed from Geometry, which does not yet know of such fractional exponents"—and it is, in view of the diagram we traditionally draw when we explain what it means to construct $Df(x)$, quite natural. The "geometrical meaning" of $D^{-1}f(x)$ is similarly direct. But when we look to $D^{\pm n}f(x)$ our geometrical intuition becomes progressively more tenuous as n increases; we find it entirely natural to abandon our geometrical representations, and to adopt a more formal, a more algorithmic mode of thought. We have only a vaguely qualitative image in mind when we contemplate the Fourier transformation procedure, and get along well enough with less than that when doing (say) Hankle transforms. My position is that it serves no useful purpose to aspire to more when it comes to the Riemann-Liouville transform.

If geometrical imagery is, for some reason, deemed essential one can, of course (in any particular case), simply *plot* the function

$$F(x, \nu) \equiv D^{-\nu} f(x)$$

When this is done it becomes natural—natural *to the eye*—to ask (since ν has become now a continuous variable) questions pertaining

$$\frac{\partial}{\partial \nu} F(x, \nu) \equiv \text{ derivative } \textit{with respect to order}$$

And having done so, it becomes natural to consider *derivatives of fractional order* with respect to order! But this is not the occasion to venture down that road of infinite regress; I am content merely to wonder what pretty country it might pass through.

I conclude with this thought: the fractional calculus is a source of analytical power, latently too valuable to be casually dismissed. It has demonstrable applicability to a rich assortment of pure and applied subject areas. But it is valuable not least because it invites—indeed, it frequently requires—one to think about old things in new ways, and to become more intimately familiar with the resources of the *ordinary* calculus. It opens doors.